WEAK SPECIFICATION AND BAIRE SPACE

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Abstract. In this paper we compute the topological entropy and we consider the question of ω-chaos for shift spaces over countable alphabets. We show that if a shift space has the *weak specification property*, then it has ω-chaos, and we show that if, in addition, it has a countably infinite alphabet, then it has infinite topological entropy.

1. Introduction

It is a common construction to take a set of symbols, say $n = \{0, \ldots, n-1\}$, with the discrete topology and consider the set of all possible infinite words over the alphabet, i.e. words of the form

$$w = w_0 w_1 w_2 \ldots$$

with each $w_i \in \{0, \ldots, n-1\}$, equipped with the product topology, along with the natural shift map, σ, defined by

$$\sigma(w) = \sigma(w_0 w_1 w_2 \ldots) = w_1 w_2 w_3 \ldots.$$

The dynamical system that arises from this alphabet and map is denoted by (n^ω, σ) with $\omega = \mathbb{N} \cup \{0\}$ and $\sigma : n^\omega \to n^\omega$ as defined above. This system is called the *full one-sided shift on n symbols* and is fundamental in the study of symbolic dynamics. While this system is simple to describe, the dynamics are indeed quite complicated. For example, it is well-known that (n^ω, σ) is chaotic in the sense of Devaney, Li and Yorke, and its topological entropy is $\log(n)$, [17] and [15].

A *subshift* of n^ω is a compact σ-invariant subset of n^ω. For $n > 1$, there are uncountably many distinct subshifts [9], and these subshifts display a wide variety of dynamics. For example, a significant body of research exists on the classification of subshifts with various types of chaos, [1], [4], [13], and [18]. In addition to chaotic dynamics, subshifts have proven to be amenable to analysis of asymptotic dynamical notions such as ω-limit sets. Indeed, we have several results related to classifying the structure of ω-limit sets of certain subshifts, [5] and [6]. Even though such dynamical systems are interesting in their own right, they have a much broader application, in that many dynamical systems on more complicated spaces can be described as quotients of subshifts. For example, Baldwin has shown that dendritic Julia sets in $\mathbb{C}$ can be written as countable-to-one quotients of certain subshifts of $(3^\omega, \sigma)$, [2] and [3]. Moreover, he showed that circular Julia sets can also be written as quotients of subshifts of $(4^\omega, \sigma)$. We have used this symbolic representation of these systems to prove results related to shadowing and classifying ω-limit sets of these Julia sets, [8] and [7]. Given that the full shift for $n > 1$ is a dynamical

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system on a topological Cantor set, it is immediate that the underlying spaces for subshifts is also totally disconnected. In light of this, it is clear that the quotients mentioned above are necessarily quite complicated, but they nonetheless allow for the analysis of these systems through the analysis of the related subshifts. One type of subshift that is particularly prominent is the family of *shifts of finite type.* It has recently been shown by Meddaugh and Good that subshifts of finite type are a fundamental building block for systems with shadowing [10].

So it is apparent that even though the family of subshifts of symbolic dynamical systems seems like quite simple mathematical objects, their utility is far-reaching as encoding sets for much more exotic dynamical systems.

In this paper we focus on symbolic dynamical systems with a countably infinite alphabet, Σ, equipped with the discrete topology, and with shift map σ acting on Σ^ω equipped with the product topology. Then (Σ^ω, σ) is the analogous shift space to (n^ω, σ). The space Σ^ω is a totally disconnected metric space, and if $\Sigma = \mathbb{N}$ then we call it *Baire Space.* Since Σ is countably infinite and has the discrete topology, Σ and Σ^ω are clearly not compact. This lack of compactness leads to many complications in the proofs that follow. The techniques that we employ are slightly different than they would be in the finite alphabet case.

In this paper we extend the results of Lampart and Oprocha [14] by showing that a closed, σ-invariant subset $\Gamma \subseteq \Sigma^\omega$ (a *subshift* of Σ^ω) with the *weak specification property (WSP)* has *ω-chaos* (both terms are defined in the next section.) We also show that if Γ has an underlying alphabet that is infinite then the system $(\Gamma, \sigma|_\Gamma)$ has infinite topological entropy.

2. Preliminaries

For the purposes of this paper, a dynamical system is a pair (X, f) where X is a metric space that may or may not be compact and $f : X \to X$ is a continuous map. We will make use of the following notions from the general theory of dynamical systems.

A dynamical system (X, f) has the *specification property* provided for every $\delta > 0$ there is some $N_\delta \in \mathbb{N}$ such that for $n \geq 2$ and for any n points $x_1, \ldots, x_n \in X$ and any sequence of natural numbers $a_1 \leq b_1 < a_2 \leq b_2 < \cdots < a_n \leq b_n$ with $a_i - b_{i-1} \geq N_\delta$ for $2 \leq i \leq n$ there is a point $x \in X$ such that $d(f^j(x), f^j(x_i)) < \delta$ for $a_i \leq j \leq b_i$ and $1 \leq i \leq n$. The map has the *weak specification property* if the above holds with $n = 2$. It is worth noting that some authors additionally require that the witnessing points for both of these properties by periodic with some proscribed period. However, for the purposes of this paper, we do not impose this condition.

The specification property and the weak version are very useful for the construction of invariant measures on dynamical systems [16, 19, 20].

The *topological entropy* of the system (X, f) is a measure of the complexity of the dynamics of the system. There are many notions of topological entropy, especially if non-compact domains are considered. For the purposes of this paper, we take the notion of topological entropy as defined by Hofer [11]. For a finite open cover $\mathcal{U}$ of X and natural number n, let $\vee^n \mathcal{U} = \{U_0 \cap f^{-1}(U_1) \cap \cdots \cap f^{1-n}(U_{n-1}) : U_i \in \mathcal{U}\}$. This is also a finite open cover of X provided that f is surjective. If we let $\mathcal{F}(X)$ denote the collection of finite open covers of X, the topological entropy of (X, f) is the number $h(f)$ given by

$$h(f) = \sup_{\mathcal{U} \in \mathcal{F}(X)} \left(\lim_{n \to \infty} \left(\frac{1}{n} \log |\vee^n \mathcal{U}| \right) \right).$$

Note, that in the event that X is compact, this is equivalent to the standard notions of topological entropy [11].

For a system (X, f), and $x \in X$, the ω-limit set of x is the set

$$\omega(x) = \bigcap_{N \in \mathbb{N}} \overline{\{f^i(x) : i \geq N\}}.$$

This set is closed and f invariant. In the event that X is compact, it also compact and nonempty. In the non-compact setting, this is not always the case.

A point $x \in X$ is *periodic* provided that there exists $n \in \mathbb{N}$ with $f^n(x) = x$. The set of periodic points is denoted $Per(f)$.

In a system (X, f), we say that a set W of cardinality at least two is ω-scrambled provided that for all $x \neq y \in W$,

(1) $\omega(x) \setminus \omega(y)$ is uncountable,
(2) $\omega(x) \cap \omega(y) \neq \varnothing$, and
(3) $\omega(x) \setminus Per(f) \neq \varnothing$.

We say that the system has ω-chaos provided that there is an uncountable ω-scrambled set.

The primary focus of this paper is on category of dynamical systems known as shift spaces. Let Σ be an alphabet, i.e. a set equipped with the discrete topology. There are two natural associated dynamical systems. The first is the full (one-sided) shift over Σ, i.e. the shift map $\sigma : \Sigma^\omega \to \Sigma^\omega$ defined by

$$\sigma(w_0 w_1 w_2 \dots) = w_1 w_2 w_3 \dots.$$

By taking Σ^ω with the product topology, this map is a continuous surjection, but if $|\Sigma| > 1$, it is not injective. The second system is the full two-sided shift over Σ, which consists of the natural shift map σ acting on the set $\Sigma^{\mathbb{Z}}$ of bi-infinite words from Σ. If $\Sigma^{\mathbb{Z}}$ is given the product topology, then σ is a homeomorphism in this setting. The results of this paper are established in the case of Σ^ω but the proofs can be easily modified for the case of $\Sigma^{\mathbb{Z}}$.

Typically it is assumed that the alphabet is finite, but in this paper we allow Σ to be countable (either finite or countably infinite.) In this context, we define a *subshift* of Σ^ω to be a closed σ-invariant subset. Note that in the more commonly studied case of Σ being finite, subshifts are taken to be compact while in the general case we do not require compactness.

In the case that $\Sigma = \mathbb{N} \cup \{0\}$, Σ^ω is the Baire space and any subshift $\Gamma \subseteq \Sigma^\omega$ is called a *countable state Markov shift.*

The product topology on Σ^ω is compatible with the following metric on Σ^ω. For $x, y \in \Sigma^\omega$, define

$$d(x, y) = \sum_{i=0}^{\infty} \frac{a_i}{2^i}$$

where

$$a_i = \begin{cases} 0, & \text{if } x_i = y_i; \\ 1, & \text{otherwise.} \end{cases}$$

Let Σ^* denote the finite length words over the alphabet Σ. Given some $w \in \Sigma^*$ let $|w|$ denote its length. Given $i, j \in \omega$ with $i < j$, define $w_{[i,j]} = w_i \dots w_j$ and $w_{[i,j)} = w_i \dots w_{j-1}$, with $w_{(i,j]}$ and $w_{(i,j)}$ defined analogously. Given a subshift $\Gamma \subset \Sigma^\omega$ and $m \in \mathbb{N}$, let the set of *m-allowed words*, $B_m(\Gamma)$, be the set of all words, w, of length m such that there is some $x \in \Gamma$ and $i \in \omega$ such that $x_{[i,i+m)} = w$. The *language* of Γ is the set

$$B(\Gamma) = \bigcup_{m \in \mathbb{N}} B_m(\Gamma)$$

of finite allowed words for Γ.

If w is a finite word of some length, say $k \in \mathbb{N}$, then the *cylinder set defined by* w is the set

$$[w] = \{x \in \Gamma : x_{[0,k)} = w\}.$$

The collection of all cylinder sets forms a basis for the topology on Γ. It is easy to see that each $[w]$ is clopen, but in general they are not compact sets. We refer to all $v \in B(\Gamma)$ with $v_{[0,k)} = w$ as the *extensions of* w and for each such v we say that w is an *initial segment* or *initial subword* of v.

A significant benefit of working in shift spaces over the full generality of dynamical systems is that there are often reasonable characterizations of complicated dynamical properties in terms of the language of the shift. The following lemma is of this flavor, allowing the WSP property to be nicely characterized.

Lemma 1. *Let Σ be a countable (finite or infinite) alphabet, and let Γ be a subshift of Σ^ω with language $B(\Gamma)$. Then Γ has the WSP if, and only if, there is some $n \in \mathbb{N}$ such that for every $u, v \in B(\Gamma)$ there is a word w of length n such that $uwv \in B(\Gamma)$.*

Proof. Suppose that Γ has the WSP. Let $n = N_{1/2} \in \mathbb{N}$ be chosen by the WSP for $\delta = \frac{1}{2}$. Let $u, v \in B(\Gamma)$. Let $s \in [u]$, and let $t \in \sigma^{-(|u|+n)}([v])$. Let $a_1 = 0$, $b_1 = |u|$ and $a_2 = |u| + n$ and $b_2 = |u| + n + |v|$. Then the point $x \in \Gamma$ that has the property that $d(\sigma^i(x), \sigma^i(s)) < \frac{1}{2}$ for $0 = a_1 \leq i \leq b_1 = |u|$ will agree with s for its first $|u|$-many symbols, i.e. $x_{[a_1,b_1)} = u$. Thus x has u as an initial segment. Moreover, $d(\sigma^i(x), \sigma^i(t)) < \frac{1}{2}$ for $|u| + n = a_2 \leq i \leq b_2 = |u| + n + |v|$ implies that in fact $x_{[a_2,b_2]} = t_{[a_2,b_2]}$. Then x has a subword, $w = x_{[b_1,a_2)}$, of length n such that uwv is an initial segment of x. So $uwv \in B(\Gamma)$.

Now, instead, suppose that there is some $n \in \mathbb{N}$ such that for every $u, v \in B(\Gamma)$ there is a word $w \in B(\Gamma)$ of length n with $uwv \in B(\Gamma)$. Let $\delta > 0$. Choose $M_\delta \in \mathbb{N}$ so that $M_\delta > n$ and so large that if $x, y \in \Gamma$ have $x_{[0,M_\delta]} = y_{[0,M_\delta]}$ then $d(x, y) < \delta$. Let $N_\delta = M_\delta + n$. Let $y, z \in \Gamma$ and let $a_1 \leq b_1 < a_2 \leq b_2$ with $a_2 - b_1 \geq N_\delta$.

Let u be the initial segment of y of length $b_1 + M_\delta$, and let v be the segment of z from position $b_1 + N_\delta$ to $b_2 + M_\delta$. Then by our assumptions there is a word w of length n with $uwv \in B(\Gamma)$. Let $x \in \Gamma$ have uwv as its initial segment. This x has the property that

$$x_{[i,i+M_\delta]} = y_{[i,i+M_\delta]}$$

for $i \in \{0, \dots, b_1\}$ specifically this will hold for all $i \in \mathbb{N}$ with $a_1 \leq i \leq b_1$. From this it follows that

$$d(\sigma^i(x), \sigma^i(y)) < \delta$$

for $i \in \mathbb{N}$ with $a_1 \leq i \leq b_1$. Moreover,

$$x_{[i,i+M_\delta]} = z_{[i,i+M_\delta]}$$

for $i \in \{b_1 + N_\delta, \dots b_2\}$, specifically this will hold for all $i \in \mathbb{N}$ with $a_2 \le i \le b_2$. So we have

$$d(\sigma^i(x), \sigma^i(z)) < \delta$$

for $i \in \mathbb{N}$ with $a_2 \le i \le b_2$. Thus (Γ, σ) has the WSP. □

For a given $k \in \mathbb{N}$, we will write $u *^{(k)} v$ to mean the concatenation of u and v with some word w of length k when the actual word is not relevant. With this notation we can re-state the lemma above as Γ has the WSP if, and only if, $\exists n \in \mathbb{N}$ such that for every $u, v \in B(\Gamma)$

$$u *^{(n)} v \in B(\Gamma).$$

We refer to the n in the previous lemma as the *WSP number* for Γ.

We would like to use this lemma to define points in Γ via infinite concatenation. For instance if $(u_i)_{i\in\mathbb{N}}$ is an infinite sequence of words in $B(\Gamma)$ we would like to be able to build a word

$$x = u_1 *^{(n)} u_2 *^{(n)} u_3 \dots\dots$$

In the case that Σ is finite there is never any difficulty in using the $*^{(n)}$ notation and defining such a point since Γ is compact in this setting and there are only finitely many words of length n that could fill in for the $*^{(n)}$. But in the case that Σ is countably infinite, we only assume that Γ is closed, and we can have infinitely many choices for the various $*^{(n)}$ words. The simple lemma below shows how to construct such a point in this case.

Lemma 2. *Let Σ be a countable alphabet, and let $\Gamma \subseteq \Sigma^\omega$ be a subshift with the WSP and WSP number n. Let $(u_i)_{i\in\mathbb{N}}$ be a sequence of finite words in $B(\Gamma)$. Then there exists a sequence of words, $(w_i)_{i\in\mathbb{N}}$, in $B(\Gamma)$ of length n and a point $x \in \Gamma$ such that*

$$x = u_1 w_1 u_2 w_2 \dots u_j w_j \dots\dots$$

Proof. By the WSP there is a word $w_0 \in B_n(\Gamma)$ such that $U_1 = u_1 w_1 u_2 \in B(\Gamma)$. Let $x_1 \in [U_1]$. Suppose that $(w_i)_{i=1}^k$ and $U_k = u_1 w_1 \dots w_k u_{k+1} \in B(\Gamma)$ have been defined, and let $x_k \in [U_k]$. Then there is some $w_{k+1} \in B_n(\Gamma)$ such that $U_k w_{k+1} u_{k+1} \in B(\Gamma)$. Let $x_{k+1} \in [U_{k+1}]$. Let $x = u_1 w_1 u_2 w_2 \dots u_j w_j \dots \dots \in \Sigma^\omega$. Since $\Gamma \subseteq \Sigma^\omega$ is closed and since $x_k \to x$ we see that $x \in \Gamma$. □

3. Entropy and ω-chaos for subshifts with the WSP

In this section we begin by considering the entropy of subshifts, Γ, of Σ^ω with Σ a countably infinite or a finite alphabet and with Γ having the WSP. We show that subshifts, Γ, with $|B_1(\Gamma)| \ge m$ and WSP number n have entropy at least $\frac{\log(m)}{n+1}$. It follows that subshifts with a countably infinite alphabet, $|B_1(\Gamma)| = |\mathbb{N}|$, have infinite entropy. We begin by showing that Γ contains a subset that is semi-conjugate to the full one-sided shift on m-symbols, m^ω, as long as $|B_1(\Gamma)| \ge m$ and Γ has the WSP.

Theorem 3. *Let Σ^ω be the full one-sided shift space over the finite or countably infinite alphabet Σ. Let Γ be a subshift of Σ^ω with the WSP, WSP number n, and $m \in \mathbb{N}$ such that $|B_1(\Gamma)| \ge m$. Then there is a set $\Omega \subseteq \Gamma$ and a semiconjugacy π from the system (Ω, σ^{n+1}) onto (m^ω, σ), the full one-sided shift on m symbols.*

Proof. Let $\{k_0, \dots, k_{m-1}\} \subseteq B_1(\Gamma)$. For every $\alpha = \alpha_0\alpha_1 \cdots \in m^\omega$, let

$$\Omega_\alpha = \{x \in \Gamma : x = k_{\alpha_0} *^{(n)} k_{\alpha_1} *^{(n)} k_{\alpha_2} \dots \text{ for some choice of the } *^{(n)} s\}.$$

Let

$$\Omega = \bigcup_{\alpha \in m^\omega} \Omega_\alpha$$

and observe that σ^{n+1} maps Ω onto itself. Define $\pi : \Omega \to m^\omega$ by $\pi(x) = \alpha$ if, and only if, $x \in \Omega_\alpha$. It is easy to see that π semiconjugates the action of σ^{n+1} on Ω with the action of σ on m^ω. □

Corollary 4. *Let Σ^ω be a full shift space over the finite or countably infinite alphabet Σ. Let Γ be a subshift of Σ^ω with the WSP, WSP number n, and $m \in \mathbb{N}$ such that $|B_1(\Gamma)| \geq m$. Then the topological entropy, $h(\sigma|_\Gamma)$, of the system (Γ, σ) is at least $\frac{\log(m)}{n+1}$.*

Proof. Let $\Omega' = \bigcup_{i \in n+1} \sigma^i(\Omega) \subseteq \Gamma$. Then σ maps Ω' onto itself, and it is well known that $h(\sigma|_\Gamma) \geq h(\sigma|_\Omega)$ since $\Omega \subseteq \Gamma$ is closed and σ-invariant. Also $(n+1)h(\sigma|_\Omega) = h(\sigma^{n+1}|_\Omega)$. Finally since (Ω, σ^{n+1}) is semiconjugate to (m^ω, σ) we see that $h(\sigma^{n+1}|_\Omega) \geq \log(m)$. The result follows. □

Theorem 5. *Let Σ^ω be a full shift space over countably infinite alphabet Σ. Let Γ be a subshift of Σ^ω with the WSP, WSP number n, and $B_1(\Gamma)$ infinite. Then $h(\sigma_\Gamma)$ is infinite.*

We now turn to the question of ω-chaos. Lampart and Oprocha proved that if Γ is a subshift of the full shift on n symbols which is not minimal and which has the WSP property, then Γ exhibits ω-chaos [14]. We extend this result to subshifts of the full shift on countably infinite alphabets.

Lemma 6. *Let Γ be a subshift of Σ^ω where Σ is a countably infinite alphabet. Assume that Γ has WSP number n and that $B_1(\Gamma)$ is not finite. For each $w \in B(\Gamma)$, $[w]$ contains uncountably many points $(p_\lambda)_{\lambda \in \mathfrak{c}}$ with the property that $\omega(p_\lambda)$ is uncountable, contains no periodic points and is disjoint from $\omega(p_\gamma)$ for every other $\gamma \in \mathfrak{c}$.*

Proof. Fix $w \in B(\Gamma)$, and let $\hat{w}$ be an extension of w with $|\hat{w}| \geq n+1$ and such that the final symbol of $\hat{w}$ occurs in $\hat{w}$ exactly once. Since $B_1(\Gamma)$ is infinite, let $k_0, k_1 \in B_1(\Gamma)$ so that neither k_0 nor k_1 appears in $\hat{w}$. Also, define $r = |\hat{w}| + 2n + 1$.

Let $(M_\lambda)_{\lambda \in \mathfrak{c}}$ be an uncountable collection of disjoint minimal sets in the full one-sided shift on two symbols, $(\{k_0, k_1\}^\omega, \sigma)$ [9]. For each $\lambda \in \mathfrak{c}$, let $\mu_\lambda \in M_\lambda$ with

$$\mu_\lambda = \mu_0^\lambda \mu_1^\lambda \mu_2^\lambda \dots\dots$$

For each $N \in \mathbb{N}$, consider the set $E_\lambda(N)$ of all words in $B_{Nr+1}(\Gamma)$ of the form

$$\mu_0^\lambda *^{(n)} \hat{w} *^{(n)} \mu_1^\lambda *^{(n)} \cdots \hat{w} *^{(n)} \mu_N^\lambda.$$

Since Γ has WSP number n, this set is nonempty and is either finite or countable. Choose a surjection $\phi_N : \mathbb{N} \to E_\lambda(N)$, and define $e_\lambda(N, k) = \phi_N(k)$.

Now, choose a function $\psi : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ such that for each $(a, b) \in \mathbb{N} \times \mathbb{N}$, $\psi^{-1}((a, b))$ is cofinal in $\mathbb{N}$.

We now choose an increasing sequence of indices i_k as follows. Choose $i_1 > 0$ such that

$$(\mu_\lambda)_{[i_1, i_1 + \pi_1(\psi(1))]} = (\mu_\lambda)_{[0, \pi_1(\psi(1))]}.$$

Once i_k has been chosen, select i_{k+1} so that $i_{k+1} > i_k + \pi_1(\psi(k))$ and so that

$$(\mu_\lambda)_{[i_{k+1}, i_{k+1}+\pi_1(\psi(k+1))]} = (\mu_\lambda)_{[0,\pi_1(\psi(k+1))]}.$$

We now define a specific point p_λ of the form

$$p_\lambda = \hat{w} *^{(n)} \mu_0^\lambda \cdots *^{(n)} \hat{w} *^{(n)} \mu_{i_1}^\lambda \cdots *^{(n)} \hat{w} *^{(n)} \mu_{i_2}^\lambda \cdots *^{(n)} \hat{w} *^{(n)} \mu_{i_3}^\lambda \cdots .$$

In particular, we choose p_λ so that

$$(p_\lambda)_{[r(1+i_k)-n, r(1+i_k)-n+\pi_1(\psi(k))]} = e_\lambda(\psi(k)).$$

Notice that, by construction, $(p_\lambda)_{[r(1+k)-n]} = \mu_k^\lambda$ and $p_\lambda \in [w]$.

Let $P_\lambda = \omega(p_\lambda)$. Since for $N \leq M$, each word of $E_\lambda(N)$ is a subword of a word of $E_\lambda(M)$, every word of $E_\lambda(N)$ occurs infinitely often in p_λ.

Now, let $\nu = \nu_0\nu_1\nu_2 \ldots \in M_\lambda$. Since Γ has WSP number n, we can find a point $t_\nu = \nu_0 *^{(n)} \hat{w} *^{(n)} \nu_1 *^{(n)} \cdots \in \Gamma$. For a fixed $N \in \mathbb{N}$, there exists $M \in \mathbb{N}$ such that $\nu_0\nu_1 \ldots \nu_N$ is a subword of $\mu_0^\lambda \mu_1^\lambda \ldots \mu_M^\lambda$, and as such we see that $\nu_0 *^{(n)} \hat{w} *^{(n)} \nu_1 *^{(n)} \cdots *^{(n)} \nu_N$ is a subword of some word in $E_\lambda(M)$, and as such occurs infinitely often in p_λ. In particular, $t_\nu \in P_\lambda$, i.e. P_λ contains representatives for each $\nu \in M_\lambda$, and thus is uncountable. A similar argument shows that, in fact, every element of P_λ is of this form, i.e. $x \in P_\lambda$ if and only if there exists $\nu = \nu_0\nu_1\nu_2 \ldots \in M_\lambda$ and $k \leq r$ with

$$x = *^{(k)} \nu_0 *^{(n)} \hat{w} *^{(n)} \nu_1 *^{(n)} \cdots \hat{w} *^{(n)} \nu_N^\lambda *^{(n)} \cdots$$

From this, it follows that $P_\lambda \cap Per(\sigma)$ is empty, as if $x \in P_\lambda \cap Per(\sigma)$, the associated $\nu \in M_\lambda$ would necessarily be periodic as well, and this is a contradiction.

Suppose for some $\lambda, \gamma \in \mathfrak{c}$ we have some $x \in P_\lambda \cap P_\gamma$. Since M_λ and M_γ are compact and disjoint, there exists a length L and finite words $v_1^\lambda, \ldots, v_s^\lambda$ and $v_1^\gamma, \ldots v_t^\gamma$ in $\{0,1\}^L$ such that

$$M_\lambda \subseteq \bigcup_{i=1}^{s} [v_s^\lambda]$$

and

$$M_\gamma \subseteq \bigcup_{i=1}^{t} [v_i^\gamma]$$

and such that $v_i^\lambda \neq v_j^\gamma$ for all $1 \leq i \leq s$ and $1 \leq j \leq t$.

Without loss of generality, we may assume that $x \in [\hat{w}]$. Since x is in the ω-limit set of p_λ and p_γ, there are increasing sequences $(n_i)_{i\in\mathbb{N}}$ and $(m_i)_{i\in\mathbb{N}}$ with $\sigma^{n_i}(p_\lambda) \to x$ and $\sigma^{m_i}(p_\gamma) \to x$. Choose $I \in \mathbb{N}$ so large that

(1) for all $i \geq I$, $\sigma^i(\mu_\lambda) \in \bigcup_{i=1}^s [v_s^\lambda]$,
(2) for all $i \geq I$, $\sigma^i(\mu_\gamma) \in \bigcup_{i=1}^t [v_i^\gamma]$, and
(3) for all $i \geq I$, $\sigma^{n_i}(p_\lambda)$ and $\sigma^{m_i}(p_\gamma)$ both agree with x for their first Lr many symbols.

It follows from choice of $\hat{w}$ that for $i \geq I$, n_i and m_i are multiples of r. For $i \geq I$, define $n_i', m_i' \in \mathbb{N}$ so that $n_i = rn_i'$ and $m_i = rm_i'$.

Then, for all $0 \leq j \leq L$ and for all $i \geq I$, we have $p_{jr+rn_i'}^\lambda = x_{jr} = p_{jr+rm_i'}^\gamma$. By the construction of p_λ and p_γ we see that $\mu_{j+n_i}^\lambda = \mu_{j+m_i}^\gamma$ for $0 \leq j \leq L$ and for all $i \geq I$. Since m_i and n_i are greater than i, this implies that some pair v_u^λ and v_w^γ are equal, a contradiction. Thus the collection $(P_\lambda)_{\lambda\in\mathfrak{c}}$ is pairwise disjoint.

$\square$

Corollary 7. *Let Γ be a subshift of Σ^ω where Σ is a countably infinite alphabet. Assume that Γ has WSP number n and that $B_1(\Gamma)$ is not finite. Then Γ is not minimal.*

Proof. For each $\lambda \in \mathfrak{c}$, the set P_λ is a closed invariant proper subsystem of Γ, and thus Γ is not minimal. □

Theorem 8. *Let Γ be a subshift of $\Sigma^\mathbb{N}$ where Σ is a countably infinite alphabet. If Γ has WSP and $B_1(\Gamma)$ is not finite, then Γ has ω-chaos.*

Proof. Let n be the WSP number for Γ. By Corollary 7, Γ is not minimal, and so we can fix $z \in \Gamma$ with $\overline{Orb(z)} \neq \Gamma$. Let $w \in B(\Gamma)$ with $[w] \cap \overline{Orb(z)} = \emptyset$. Let $(p_\lambda)_{\lambda\in\mathfrak{c}}$ be points in $[w]$ guaranteed by Lemma 6. Let $r_1 < r_2 < \cdots$ be an increasing sequence in $\mathbb{N}$ and for each $\lambda \in \mathfrak{c}$, let

$$q_\lambda = z_{[0,r_1]} *^{(n)} (p_\lambda)_{[0,r_1]} *^{(n)} z_{[0,r_2]} *^{(n)} (p_\lambda)_{[0,r_2]} \cdots .$$

It is immediate that $\omega(q_\lambda)$ contains the point z and the set P_λ. In particular, for each pair $\lambda, \eta \in \mathfrak{c}$, $\omega(q_\lambda) \cap \omega(q_\eta) \neq \varnothing$, and $\omega(q_\lambda) \cap Per(\sigma)$ is uncountable.

To demonstrate that Γ has ω-chaos, all that remains to be shown is that for each pair $\gamma \neq \lambda \in \mathfrak{c}$, the set $\omega(q_\lambda) \setminus \omega(q_\gamma)$ is uncountable. Towards this end, for each $\lambda \neq \gamma \in \mathfrak{c}$, and suppose $y \in P_\lambda \cap \omega(q_\gamma)$. Without loss of generality, we may also assume that $y \in [w]$.

For each $k \in \mathbb{N}$, the initial segment $y_{[0,k)}$ of y occurs infinitely often in q_γ. Since w is an initial segment of y, and $[w] \cap \overline{Orb(z)} = \emptyset$, $y_{[0,k)}$ is of one of the following forms.

(1) $y_{[0,k)} = z_{[r_l-t,r_l]} *^{(n)} p^\gamma_{[0,k+1-n-t)}$ for some $t < |w|$ and some $l \in \mathbb{N}$,
(2) $y_{[0,k)} = *^{(s)} p^\gamma_{[0,k-s)}$ for some $s \leq n$, or
(3) $y_{[0,k)} = p^\gamma_{[u,u+k)}$ for some $u \geq 0$.

In case (3), we see that either $y = \sigma^u(p_\gamma)$ in the event that there is some u that witnesses this for infinitely many k, or else each initial segment of y occurs infinitely often in p_γ. In the former case, we see that $P_\gamma = \omega(y) \subseteq P_\lambda$ and in the latter, we see that $y \in \omega(p_\gamma) = P_\gamma$. In either case, $P_\lambda \cap P_\gamma \neq \varnothing$ and therefore $\lambda = \gamma$, a contradiction.

In cases (1) and (2), there is some fixed $m \leq |w| + n$ with $y_{[m,k)} = p^\gamma_{[0,k-m)}$ for infinitely many k, and so we see that $\sigma^m(y) = p_\gamma$. It follows that $P_\gamma = \omega(y) \subseteq P_\lambda$, i.e. $P_\gamma \cap P_\lambda \neq \varnothing$ and so $\gamma = \lambda$, a contradiction.

□

Note that in the preceding, the assumption that $B_1(\Gamma)$ is not finite combined with the WSP property actually implies the non-minimality that was a necessary assumption in the theorem of Lampart and Oprocha [14]. In the event that Γ is a subshift a shift space with countably infinite alphabet with $B_1(\Gamma)$ finite, then Γ can be considered as a subshift on a finite alphabet, and as such, the results of Lampart and Oprocha can be applied.

Corollary 9. *Let Γ be a subshift of $\Sigma^\mathbb{N}$ where Σ is a countable alphabet. If Γ is not minimal and has the WSP property, then Γ has ω-chaos.*

These results are a natural extension of the earlier result due to Lampart and Oprocha in the finite aphabet case, [14]. In that paper they showed that any subshift of a finite alphabet shift with the WSP must have ω-chaos, and they asked if the result holds in general for a compact metric space, X, with continuous map $f : X \to X$ with the WSP. Hunter and Raines proved that if $f : X \to X$ with X compact and f continuous with the *specification property* and expansivity near a fixed point, then the system (X, f) has ω-chaos, [12].

Since shift spaces over countably infinite alphabets are a natural analogue to non-compact dynamical systems, it is then a natural question to ask if $f : Y \to Y$ is a continuous map on a (non-compact) metric space Y with the specification property and expansivity near a fixed point, does the system (Y, f) have ω-chaos?

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