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SHADOWING IS GENERIC ON DENDRITES

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Abstract. We show that shadowing is a generic property for continuous maps on dendrites.

1. **Introduction.** One of the most well-studied properties in the theory of topological dynamical systems is *shadowing* or the *pseudo-orbit tracing property* that was introduced independently by Anosov, [1], and Bowen, [2]. Let (X, d) be a compact metric space and $f : X \to X$ continuous. For $\delta > 0$, a sequence $\langle x_i \rangle_{i \in \mathbb{N}}$ is a δ-*pseudo-orbit* provided $d(f(x_i), x_{i+1}) < \delta$ for all i. For $\varepsilon > 0$, a point $z \in X$ is said to ε-*shadow* a pseudo-orbit $\langle x_i \rangle_{i \in \mathbb{N}}$ provided $d(f^i(z), x_i) < \varepsilon$ for all i. We say that the map f has *shadowing* or the *pseudo-orbit tracing property* if for every $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit is ε-shadowed by some point. Computer approximations of dynamical systems by necessity usually deal with pseudo-orbits rather than real orbits. If a system has shadowing, then we can be sure that every pseudo-orbit a computer generates is followed by an actual orbit.

Given a compact metric space (X, d), let $\mathcal{C}(X)$ denote the space of continuous self-maps of X, with the topology induced by the supremum metric

$$\rho(f, g) = \max_{x \in X} d(f(x), g(x)).$$

This metric is complete on $\mathcal{C}(X)$. The topology it induces coincides with both the compact-open topology and the topology of uniform convergence.

For our purposes, a dynamical system consists of a compact metric space X and a continuous map $f : X \to X$. If X is given in advance, then we may think of a dynamical system simply as a point of $\mathcal{C}(X)$. It is in this sense that we speak of dynamical properties as being "generic" for a space X: it means that the set of all $f \in \mathcal{C}(X)$ with that property is co-meager.

The question of the genericity of shadowing has been studied for some time, but usually in the context of the space of homeomorphisms on a manifold with the C^0

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topology. Yano showed that shadowing is generic for homeomorphisms of the unit circle, [12], and Odani proved that shadowing is generic for homeomorphisms on smooth manifolds with dimension at most three, [10]. Pilyugin and Plamenevska extended this to homeomorphisms on compact manifolds without boundary but with a handle decomposition, [11].

In contrast to these results we consider the space of all continuous functions, rather than just homeomorphisms, on a dendrite D. A *dendrite* is a compact, locally connected, uniquely arcwise connected, metric space; roughly, it is a compact tree-like space, where the tree may branch infinitely often, or even have a dense set of branching points. These spaces arise frequently in the study of Julia sets on the complex plane, [3]. Our main theorem is that the shadowing property is generic for continuous maps on dendrites:

Main Theorem. *Let D be a dendrite and let $\mathcal{C}(D)$ denote the space of all continuous self-maps of D. The set of all $f : D \to D$ with the shadowing property is a co-meager subset of $\mathcal{C}(D)$.*

The analogous result was established by Mizera for continuous maps on $[0, 1]$ and the unit circle, [9]. Recently this type of result was also established for compact manifolds by Mazur and Oprocha, [6], and also for surjections on manifolds that admit a decomposition by Kościelniak, Mazur, Oprocha, and Pilarczyk, [8]. Using different techniques, Bernardes and Darji, [4], established that shadowing is generic for homeomorphisms of the Cantor space. See also [5] and [7] for further results along these lines. 1 We prove our main result in Section 3 after developing the necessary preliminaries in Section 2.

2. **Preliminaries.** Let (X, d) be a compact metric space and $f : X \to X$ a continuous map. For $x \in X$, the *orbit of* x is the sequence $\langle f^i(x)\rangle_{i\in\mathbb{N}}$.

For $\varepsilon > 0$, an *ε-pseudo-orbit* is a sequence $\langle x_i\rangle_{i\in\mathbb{N}}$ satisfying $d(f(x_i), x_{i+1}) < \varepsilon$ for all $i \in \mathbb{N}$. A map $f : X \to X$ has *shadowing* provided that for all $\varepsilon > 0$ there exists a $\delta > 0$ such that for every δ-pseudo-orbit $\langle x_i\rangle_{i\in\mathbb{N}}$ there exists an orbit $\langle f^i(z)\rangle_{i\in\mathbb{N}}$ satisfying

$$d(x_i, f^i(z)) < \varepsilon.$$

We say that the orbit $\langle f^i(z)\rangle_{i\in\mathbb{N}}$ *ε-shadows* the δ-pseudo-orbit $\langle x_i\rangle_{i\in\mathbb{N}}$.

As mentioned above, dendrites are uniquely arcwise connected. Without loss of generality, the metric d on a dendrite D can be assumed to be a "taxicab metric": i.e., given points $x, y, z \in D$, if y belongs to the arc from x to z, then $d(x, z) = d(x, y) + d(y, z)$.

A dendrite D has two special types of points. An *endpoint* is a point $x \in D$ such that $D \setminus \{x\}$ is connected. A *branchpoint* is a point $x \in D$ such that $D \setminus \{x\}$ has more than two components. In a typical dendrite, both the set of endpoints and the set of branchpoints may be dense. If $x, y \in D$ then the unique arc A between x and y is denoted by $[x, y]$, and we denote $[x, y] \setminus \{x, y\}$ by (x, y).

If x, y are points of some dendrite D, then any $z \in (x, y)$ is not an endpoint, and in particular $D \setminus \{z\}$ is disconnected. This implies that every connected subset of D is uniquely arcwise connected. We will use this fact frequently, and often without comment, in the next section.

Suppose D is a dendrite and fix $x, y \in D$. Suppose that $g(0) = x$ and $g(1) = y$, but g is not defined on $(0, 1)$. In this situation, g may be *extended linearly* between

0 and 1, meaning that for $p \in (0,1)$, we put $g(p) = z$, where z is the unique point of $[x, y]$ such that

$$d(x, z) = p \cdot d(x, y).$$

If K is a compact, connected subset of a dendrite D and $x \in D$, then there is a unique point $\pi_K(x) \in K$ that is the closest to x. We call the arc $[x, \pi_K(x)]$ the *shortest arc from x to K*. Notice that $K \cap [x, \pi_K(x)] = \{\pi_K(x)\}$. Also notice that if $x \in K$ then $\pi_K(x) = x$ and the shortest arc from x to K is the degenerate arc $\{x\}$. Extending this a bit further, observe that if K_1 and K_2 are compact connected subsets of a dendrite D then there is a unique shortest arc from K_1 to K_2.

Let $\mathcal{U} = \{U_1, U_2, \dots, U_k\}$ be an open cover of a dendrite D. We say that $\mathcal{U}$ is *taut* provided $U_i \setminus \bigcup_{j \neq i} U_j$ has non-empty interior for every $i \leq k$. Clearly every open cover of D can be refined to a taut open cover.

3. **Maps of dendrites.** In this section we prove our main theorem. Most of the proof will be broken up into a sequence of smaller propositions and lemmas.

Let D be a dendrite. The strategy of the proof is as follows. For each $n \in \mathbb{N}$, let $\mathcal{R}_n$ denote the set of all $f \in \mathcal{C}(D)$ such that for some $\delta > 0$, every δ-pseudo-orbit is $\frac{1}{n}$-shadowed. We will show that each $\mathcal{R}_n$ contains a dense open set. This implies that the set $\mathcal{R} = \bigcap_{n \in \mathbb{N}} \mathcal{R}_n$ contains a dense G_δ-set in $\mathcal{C}(D)$. The functions in $\mathcal{R}$ are precisely those with shadowing, so this proves the theorem.

The difficulty lies in proving that each $\mathcal{R}_n$ contains a dense open subset of $\mathcal{C}(D)$. To do this, we will find, for every $n \in \mathbb{N}$, every $f \in \mathcal{C}(D)$, and every $\varepsilon > 0$, a map $g \in B_\varepsilon(f)$ and a $\gamma > 0$ such that

1. $B_\gamma(g) \subseteq B_\varepsilon(f)$,
2. if $h \in B_\gamma(g)$ then every γ-pseudo-orbit of h is $\frac{1}{n}$-shadowed, therefore
3. $B_\gamma(g) \subseteq \mathcal{R}_n$.

It follows that the interior of $\mathcal{R}_n$ is dense in $\mathcal{C}(D)$.

The definition of g takes place in four stages. At each stage we work with a different subspace of D:

$$A \subseteq S \subseteq T \subseteq D.$$

These spaces will be increasingly accurate, and increasingly complex, approximations to D.

The smallest space A is just a disjoint collection of arcs. Topologically, A is a very crude approximation to D; however, in a sense to be made precise soon, we will ensure that $g|_A$ contains enough information about g to capture all possible patterns of γ-pseudo-orbits. The next subspace S is a union of disjoint trees: roughly, each piece of S connects some collection of the arcs comprising A that we wish to consider "close" to each other. T is a single tree patching together all the various smaller trees comprising S, and giving a very good approximation to the structure of D.

Our plan is to define the map g first on A, and then to extend it in turn to S, to T, and finally to all of D. After defining A below, we will define $g|_A$ (which we call g_0) before defining S or T. Similarly, the definition of $g|_S$ (which we call g_1) will precede our definition of T, and the definition of $g|_T$ (which we call g_2) will precede our definition of g. Hopefully, this process of extending g piece by piece will give the reader a sense of where the proof is headed as it unfolds.

Fix $n \in \mathbb{N}$. Before defining A, let us make precise the idea that a given $f \in \mathcal{C}(D)$ imposes certain restrictions on the possible paths of a pseudo-orbit.

Let $0 < \varepsilon < \frac{1}{n}$ and let $\mathcal{U} = \{U_1, \ldots, U_k\}$ be a taut open cover of D such that

$$\max_{1 \leq i \leq k} \{\operatorname{diam}(U_i), \operatorname{diam}(f(U_i))\} < \frac{\varepsilon}{2},$$

and such that each U_i is connected. For each $1 \leq i \leq k$ let

$$\phi(i) = \left\{m : f\left(\overline{U}_i\right) \cap \overline{U}_m \neq \emptyset\right\}.$$

This generates a directed graph Φ on the vertices $\{1, \ldots, k\}$, where i is connected to m if and only if $m \in \phi(i)$. Walks through Φ correspond to possible patterns for δ-pseudo-orbits when δ is sufficiently small. We will construct g so that it has the same pseudo-orbit structure as f (i.e., it imposes the same graph Φ on $\mathcal{U}$), but so that it is also capable of shadowing each of these pseudo-orbits.

Lemma 3.1. *For each $i \leq k$, $\bigcup_{m \in \phi(i)} U_m$ is connected.*

Proof. Let $i \leq k$. For each $m \in \phi(i)$, consider the set $W_m = U_m \cup \left(\overline{U}_m \cap f\left(\overline{U}_i\right)\right)$ and observe that

$$f\left(\overline{U_i}\right) \cup \bigcup_{m \in \phi(i)} W_m = f\left(\overline{U_i}\right) \cup \bigcup_{m \in \phi(i)} U_m = \bigcup_{m \in \phi(i)} U_m.$$

By assumption, U_i is connected, which implies $f\left(\overline{U}_i\right)$ is also connected. Each W_m is connected (because U_m is connected and $U_m \subseteq W_m \subseteq \overline{U}_m$) and meets $f\left(\overline{U}_i\right)$ (because $m \in \phi(i)$). It follows that $f\left(\overline{U}_i\right) \cup \bigcup_{m \in \phi(i)} W_m = \bigcup_{m \in \phi(i)} U_m$ is connected. □

We now proceed to the definition of A. For each $1 \leq i \leq k$, order

$$\phi(i) = \{m_1 < \cdots < m_{\ell_i}\}.$$

For each $m \in \phi(i)$ let $A_{i,m}$ be a non-degenerate arc in the interior of $U_i \setminus \bigcup_{j \neq i} U_j$, or, equivalently, in $U_i \setminus \bigcup_{j \neq i} \overline{U}_j$, such that

1. $\bigcup_{m \in \phi(i)} A_{i,m}$ is contained in a single connected component of $U_i \setminus \bigcup_{j \neq i} \overline{U}_j$,
2. no $A_{i,m}$ contains an endpoint of D, and
3. the collection $\{A_{i,m} : 1 \leq i \leq k \text{ and } m \in \phi(i)\}$ is pairwise disjoint.

Let V_i denote the connected component of $U_i \setminus \bigcup_{j \neq i} \overline{U}_j$ that contains $\bigcup_{m \in \phi(i)} A_{i,m}$. Let $A = \bigcup_{i \leq k} \bigcup_{m \in \phi(i)} A_{i,m}$.

Next we define $g_0 : A \to D$ (recall that eventually we will have $g|_A = g_0$). Roughly, we will define a collection of maps $g_{i,m}$, for $i \leq k$ and $m \in \phi(i)$, that will map each arc $A_{i,m}$ across every arc $A_{m,j}$ with $j \in \phi(m)$. Thus for each possible path through Φ, we will have a sequence of arcs following that path.

The following lemma asserts that we can do exactly this, and moreover we can do it in such a way that this property is robust under small perturbations.

Lemma 3.2. *Let V be an open connected subset of the dendrite D. Let $A_1, A_2, \ldots, A_\ell$ be pairwise disjoint arcs in V such that no endpoint of any A_i is an endpoint of D. There exists a map $g : [0,1] \to D$ and $\delta > 0$ such that $A_i \subseteq g([0,1]) \subseteq V$ for all $i \leq \ell$, and for all maps $h : [0,1] \to D$ with $\rho(g,h) < \delta$, $A_i \subseteq h([0,1]) \subseteq V$ for all $i \leq \ell$.*

Proof. Choose points,

$$q_1, \ldots q_{2\ell} \in V \setminus \left(\bigcup_{i=1}^{\ell} A_i\right)$$

such that $A_i \subseteq [q_{2i-1}, q_{2i}]$ for $1 \leq i \leq \ell$, and such that none of the q_i's are endpoints of D. Define a map g from $[0,1]$ to D by first mapping

$$\frac{i-1}{2\ell-1} \to q_i$$

for $0 \leq i \leq \ell$, and then extending linearly between these points.

Because V is connected, $g([0,1]) \subseteq V$. Let $\delta > 0$ be chosen so small that

1. for every $y \in g([0,1])$, $B_\delta(y) \subseteq V$, and
2. $B_\delta(q_i) \cap \bigcup_{j=1}^{\ell} A_j = \emptyset$ for every $i \leq 2\ell$.

Let $h : [0,1] \to D$ such that

$$\rho(g,h) < \delta.$$

Let $r_i = h\left(\frac{i-1}{2\ell-1}\right)$ for each $1 \leq i \leq 2\ell$: then $d(r_i, q_i) < \delta$, so that $A_i \subseteq [r_{2i-1}, r_{2i}]$, for each $1 \leq i \leq \ell$. Thus $h([0,1]) \supseteq A_i$ for all $1 \leq i \leq \ell$. Furthermore, $h([0,1]) \subseteq B_\delta(g([0,1])) \subseteq V$. $\square$

Using Lemma 3.2, we may find, for each $1 \leq i \leq k$ and each $m \in \phi(i)$, some $g_{i,m} : A_{i,m} \to V_m$ and some $\delta_{i,m} > 0$ such that

1. $g_{i,m}(A_{i,m}) \supseteq \bigcup_{j \in \phi(m)} A_{m,j}$, and
2. if $h : A_{i,m} \to D$ is continuous with $\rho(g_{i,m}, h) < \delta_{i,m}$ then

$$\bigcup_{j \in \phi(m)} A_{m,j} \subseteq h(A_{i,m}) \subseteq V_m.$$

Define $g_0 : A \to D$ such that $g_0|_{A_{i,m}} = g_{i,m}$ for each $i \leq k$ and $m \in \phi(i)$; this is well-defined because the $g_{i,m}$ have pairwise disjoint domains. Let

$$\delta = \min\{\delta_{i,m} : 1 \leq i \leq k,\ m \in \phi(i)\}.$$

For every walk through Φ, there is a point $x \in A$ whose g_0-orbit follows it. Since walks through Φ are meant to capture all possible pseudo-orbit patterns, this feature of g_0 is what will ensure g has shadowing. In other words, we plan to ensure that every pseudo-orbit in (D, g) is shadowed already by a point in (A, g_0). In order for this to work, the extension of g_0 to D must not introduce any new pseudo-orbit patterns. Thus, let us proceed to extend g_0 carefully.

For each $1 \leq i \leq k$, we now construct an arcwise connected tree $S_i \subseteq U_i$ containing all of the $A_{i,m}$. These S_i will be the components of S. Fix $1 \leq i \leq k$. S_i is constructed recursively in ℓ_i steps. Roughly, we are piecing together a tree from the $A_{i,m}$, and each step of the recursion consists of attaching another one of the $A_{i,m}$ to the part of the tree constructed so far.

To begin, let $D_1^i = \bigcup_{m \in \phi(i)} A_{i,m}$. For the recursive step, suppose we have constructed D_{j-1}^i for some $1 < j \leq \ell_i$, and that all of the $A_{i,m_{j'}}$, $j' < j$, lie in a single arc component of D_{j-1}^i, say B_{j-1}. If $A_{i,m_j} \subseteq B_{j-1}$, then set $C_j^i = \emptyset$. Otherwise, let $C_j^i = [c_{i,j}^-, c_{i,j}^+]$ be the shortest arc between B_{j-1} and A_{i,m_j}, with $c_{i,j}^- \in B_{j-1}$ and $c_{i,j}^+ \in A_{i,m_j}$. Let

$$D_j^i = D_{j-1}^i \cup C_j^i.$$

Finally, let $S_i = D_{\ell_i}^i$ and $S = \bigcup_{i \leq k} S_i$.

Lemma 3.3. *For each $1 \leq i \leq k$,*

1. *$S_i \subseteq V_i \subseteq U_i$.*
2. *for each $m_j \in \phi(i)$, if $C_j^i \neq \emptyset$ then $C_j^i \cap D_{j-1}^i = \{c_{i,j}^-, c_{i,j}^+\}$.*

Proof. Because V_i is uniquely arcwise connected, an easy induction shows that $D^i_j \subseteq V_i$ for all $j \leq \ell_i$. This proves (1), and (2) follows immediately from the above construction. $\square$

Now that S is defined, our next goal is to extend g_0 from A to S. Fix S_i with $1 \leq i \leq k$. Following the recursive definition of S_i, we will provide a recursive definition of g_1 on S_i.

To begin, set g_1 equal to g_0 on $D^i_1 = A \cap S_i$. For the recursive step, suppose g_1 has been defined already on D^i_{j-1} for some $j \leq \ell_i$, but has not yet been defined on any point of $S_i \setminus D^i_{j-1}$. If $C^i_j = \emptyset$ then there is nothing to do. Otherwise, by part (2) of Lemma 3.3, g_1 is defined on $c^-_{i,j}$ and $c^+_{i,j}$ but on no other points of C^i_j. In this case we define g_1 on $(c^-_{i,j}, c^+_{i,j})$ by extending it linearly between $c^-_{i,j}$ and $c^+_{i,j}$.

This defines g_1 on S_i for each $i \leq k$. The S_i are pairwise disjoint by part (1) of Lemma 3.3, so we have defined g_1 on S.

Proposition 1. *For each* $1 \leq i \leq k$,

$$g_1(S_i) \subseteq \bigcup_{m \in \phi(i)} U_m.$$

Proof. We prove by induction on j that $g_1(D^i_j) \subseteq \bigcup_{m\in\phi(i)} U_m$ for every $j \leq \ell_i$. This is sufficient, because $S_i = D^i_{\ell_i}$.

For the base case $j = 1$, we have $D^i_1 \cap S_i = \bigcup_{m\in\phi(i)} A_{i,m}$. For each $m \in \phi(i)$,

$$g_1(A_{i,m}) = g_0(A_{i,m}) \subseteq V_m \subseteq U_m,$$

so that $g_1(D^i_1) \subseteq \bigcup_{m\in\phi(i)} U_m$ as desired.

For the inductive step, assume $g_1(D^i_{j-1}) \subseteq \bigcup_{m\in\phi(i)} U_m$. If $C^i_j = \emptyset$, then there is nothing to prove. If not, then, by part (2) of Lemma 3.3 and the inductive hypothesis,

$$g_1(c^-_{i,j}), g_1(c^+_{i,j}) \in g_1(D^i_{j-1}) \subseteq \bigcup_{m\in\phi(i)} U_m.$$

By Lemma 3.1 and the fact that D is uniquely arcwise connected,

$$[g_1(c^-_{i,j}), g_1(c^+_{i,j})] \subseteq \bigcup_{m\in\phi(i)} U_m.$$

By the definition of g_1,

$$g_1(C^i_j) \subseteq \bigcup_{m\in\phi(i)} U_m,$$

so that $g_1(D^i_j) = g_1(D^i_{j-1}) \cup g_1(C^i_j) \subseteq \bigcup_{m\in\phi(i)} U_m$ as desired. $\square$

Next we construct the tree T by connecting all the various components of S. The definition is recursive, and is essentially identical to the definition of S_i from $A \cap V_i$.

To begin, let $F_1 = S$. For the recursive step, suppose we have constructed F_{i-1} for some $1 < i \leq k$, and that all of the S_j, $j < i$, lie in a single arc component of F_{i-1}, say G_{i-1}. If $S_i \subseteq G_{i-1}$, then set $E_i = \emptyset$. Otherwise, let $E_i = [e^-_i, e^+_i]$ be the shortest arc between G_{i-1} and S_i, with $e^-_i \in G_{i-1}$ and $e^+_i \in S_i$. Let

$$F_i = F_{i-1} \cup E_i.$$

Finally, let $T = F_k$.

The following lemmas will aid us in defining $g_2 : T \to D$.

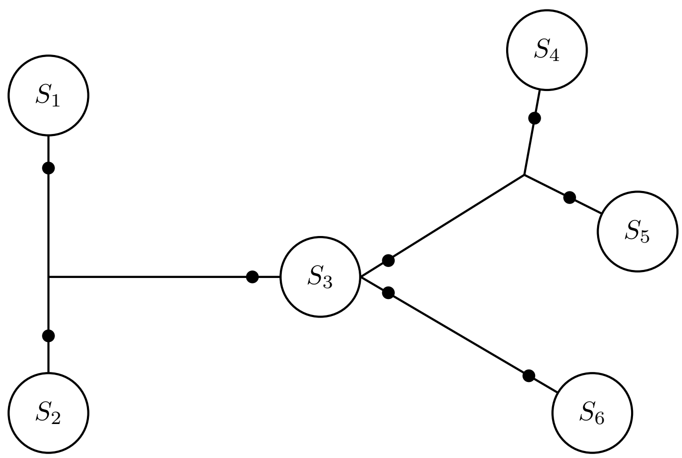
Lemma 3.4. *For each* $1 \leq i \leq k$, *if* $E_i \neq \emptyset$ *then* $E_i \cap F_{i-1} = \{e_i^-, e_i^+\}$.

Proof. This follows immediately from the above construction. □

Lemma 3.5. *There is a finite* $Z \subseteq T \setminus S$ *such that, for every* $1 \leq i \leq k$, *if* K_i *denotes the connected component of* $T \setminus Z$ *containing* S_i, *then*

1. $K_i \subseteq V_i$, *and*
2. $K_i \setminus S_i$ *is a finite union of pairwise disjoint intervals, each of the form* (s, z), *with* $s \in S_i$ *and* $z \in Z$.

The idea behind Lemma 3.5 is that we may find a finite set Z that fences off each S_i from the rest of T. A picture of (one possible version of) T and Z is shown below.



Proof of Lemma 3.5. We will construct the set Z by recursion. First, pick $\eta > 0$ small enough that, for every $1 \leq i \leq k$,

1. $\overline{B_\eta(S_i)} \subseteq V_i$,
2. for every $1 \leq j \leq k$, if $E_j \cap S_i = \emptyset$, then $\overline{B_\eta(S_i)} \cap E_j = \emptyset$.

To begin, let $Z_0 = \emptyset$. For the recursive step, we have two cases. If $E_i = \emptyset$, then do nothing: set $Z_i = Z_{i-1}$. Otherwise, we have $E_i = [e_i^-, e_i^+]$ for some $e_i^- \in F_{i-1}$ and $e_i^+ \in S_i$. In this case, let z_i^+ be the unique point of E_i such that $d(z_i^+, e_i^+) = \eta$ (uniqueness follows from the fact that we are using a taxicab metric on D). If $e_i^- \notin S$, then let $Z_i = Z_{i-1} \cup \{z_i^+\}$. If $e_i^- \in S$, then let z_i^- be the unique point of E_i such that $d(e_i^-, z_i^-) = \eta$, and let $Z_i = Z_{i-1} \cup \{z_i^+, z_i^-\}$. Finally, let $Z = Z_k$.

To prove that Z has the required properties, we use induction. Specifically, by induction on j, we show that, if K_i^j denotes the connected component of $F_j \setminus Z_j$ containing S_i then, for every $1 \leq i \leq k$,

1. $K_i^j \subseteq V_i$, and
2. $K_i^j \setminus S_i$ is a finite union of pairwise disjoint intervals, each of the form (s, z), with $s \in S_i$ and $z \in Z$.

The base case is true by part (1) of Lemma 3.3. The inductive step follows easily from Lemma 3.4 and our choice of $z_i^{\pm}$. As $F_k = T$, this completes the proof of the lemma. □

We are now ready to define $g_2 : T \to D$. For each $1 \leq i \leq k$, let K_i denote the connected component of S_i in $T \setminus Z$. The definition of g_2 is piecewise, where we view T as divided into three pieces: S, $T \setminus \bigcup_{i \leq k} K_i$, and $\bigcup_{i \leq k}(K_i \setminus S_i)$.

If $x \in S$, let $g_2(x) = g_1(x)$. If $x \in T \setminus \bigcup_{i \leq k} K_i$, let $g_2(x) = f(x)$. If $x \in \bigcup_{i \leq k}(K_i \setminus S_i)$, then $x \in (s, z)$ where $z \in Z$ and s is the point in S_i that is closest to z. On each such interval, define g_2 on (s, z) by extending it linearly between s and z (where it has already been defined).

Proposition 2. *For each $1 \leq i \leq n$,*

$$g_2\left(\overline{U}_i \cap T\right) \subseteq \bigcup_{m \in \phi(i)} U_m.$$

Proof. Let $x \in \overline{U}_i \cap T$. We have three cases:

If $x \in S_i$, then $g_2(x) = g_1(x)$, and $g_1(x) \in \bigcup_{m \in \phi(i)} U_m$ by Proposition 1.

If x is in $T \setminus \bigcup_{i \leq k} K_i$, then $g_2(x) = f(x)$. There is some $U_m \in \mathcal{U}$ containing $f(x)$, and $m \in \phi(i)$ by the definition of ϕ. Thus $g_2(x) \in \bigcup_{m \in \phi(i)} U_m$.

If $x \in \bigcup_{i \leq k}(K_i \setminus S_i)$, then x is contained in an interval of the form (s, z), where $s \in S$ and $z \in Z$. By definition, $g_2(x) \in [g_2(s), g_2(z)]$, and it is already established that $g_2(s)$ and $g_2(z)$ are in $\bigcup_{m \in \phi(i)} U_m$. By Lemma 3.1, $g_2(x) \in \bigcup_{m \in \phi(i)} U_m$ as well. □

Finally, we are ready to define $g : D \to D$. Define g so that $g|_T = g_2$, and if $x \in D \setminus T$ then $g(x) = g_2 \circ \pi_T(x)$.

It remains to show that this map g has the required properties. First we check that g imposes the same pseudo-orbit pattern on $\mathcal{U}$ that f does:

Proposition 3. *For each $1 \leq i \leq k$, $g\left(\overline{U}_i\right) \subseteq \bigcup_{m \in \phi(i)} U_m$.*

Proof. Let $x \in \overline{U}_i$, and let $[x, t]$ be the shortest path from x to t, where $t = \pi_T(x)$. Because $\overline{U}_i$ is arcwise connected, and because D is uniquely arcwise connected, we must have $[x, t] \subseteq \overline{U}_i$. Then $g(x) = g_2(t) \in \bigcup_{m \in \phi(i)} U_m$ by Proposition 2. □

Next we check that $g \in B_\varepsilon(f)$:

Proposition 4. $\rho(f, g) < \varepsilon$.

Proof. Let $x \in D$, and fix $1 \leq i \leq k$ with $x \in U_i$. By Proposition 3 there is some $m \in \phi(i)$ such that $g(x) \in U_m$. Furthermore, $f(x) \in f(U_i)$ and $f\left(\overline{U}_i\right) \cap \overline{U}_m \neq \emptyset$. By our choice of the cover $\mathcal{U}$,

$$d(f(x), g(x)) \leq \operatorname{diam}\left(f\left(\overline{U}_i\right)\right) + \operatorname{diam}\left(\overline{U}_m\right) \leq \varepsilon.$$

As x was arbitrary, it follows that $\rho(f, g) < \varepsilon$. □

Next, as promised, we find some $\gamma > 0$ such that

1. $B_\gamma(g) \subseteq B_\varepsilon(f)$;
2. if $h \in B_\gamma(g)$ then every γ pseudo-orbit of h is $\frac{1}{n}$-shadowed; therefore
3. $B_\gamma(g) \subseteq \mathcal{R}_n$.

Fix $1 \leq i \leq k$. Because

$$g(\overline{U}_i) \subseteq \bigcup_{m \in \phi(i)} U_m$$

and $g(\overline{U}_i)$ is compact, there is some $\lambda_i > 0$ such that for every $x \in \overline{U}_i$,

$$B_{\lambda_i}(g(x)) \subseteq \bigcup_{m\in\phi(i)} U_m.$$

Let $\lambda = \min\{\lambda_i : 1 \le i \le k\}$, and let

$$\gamma = \min\left\{ \varepsilon - \rho(f,g)\,,\ \frac{\lambda}{2}\,,\ \delta \right\}.$$

Because $\gamma \le \varepsilon - \rho(f,g)$, we automatically have $B_\gamma(g) \subseteq B_\varepsilon(f)$. It remains to show that for every $h \in B_\gamma(g)$, every γ-pseudo-orbit of h is $\frac{1}{n}$-shadowed by an orbit of h.

Lemma 3.6. *If $h \in B_\gamma(g)$, then, for every $1 \le i \le k$,*

1. *$h(A_{i,m}) \supseteq \bigcup_{j\in\phi(m)} A_{m,j}$ for every $m \in \phi(i)$.*
2. *$h(\overline{U}_i) \subseteq \bigcup_{m\in\phi(i)} U_m$.*

Proof. (1) If $\rho(g,h) < \gamma$, then $\rho(g|_A, h|_A) < \delta$. But $g|_A = g_0$, and by our choice of g_0 and δ, $\rho(g_0, h|_A) < \delta$ implies $h(A_{i,m}) = h|_A(A_{i,m}) \supseteq \bigcup_{j\in\phi(m)} A_{m,j}$.

(2) Suppose $\rho(g,h) < \gamma$ and let $x \in \overline{U}_i$. We have $d(g(x), h(x)) < \gamma \le \frac{\lambda}{2}$, so that $h(x) \in B_\lambda(g(x)) \subseteq \bigcup_{m\in\phi(i)} U_m$ by our choice of λ. □

We may interpret the previous lemma as asserting that for every $h \in B_\gamma(g)$, for any walk through Φ there is a sequence of arcs that, when acted on by h, follow that walk through Φ. The next lemma asserts formally that any γ-pseudo-orbit of h is described by a walk through Φ:

Lemma 3.7. *Suppose $h \in B_\gamma(g)$, and suppose $\langle x_j \rangle$ is a γ-pseudo-orbit for h. If $x_j \in U_i$, then $x_{j+1} \in \bigcup_{m\in\phi(i)} U_m$.*

Proof. Fix $h \in B_\gamma(g)$, and a γ-pseudo-orbit for h, $\langle x_j \rangle$. Suppose $x_j \in U_i$. Then $d(h(x_j), g(x_j)) < \gamma < \frac{\lambda}{2}$ (because $\rho(g,h) < \gamma$), and $d(x_{j+1}, h(x_j)) < \gamma < \frac{\lambda}{2}$ (because $\langle x_j \rangle$ is a γ-pseudo-orbit for h). Thus

$$x_{j+1} \in B_\lambda(g(x_j)) \subseteq \bigcup_{m\in\phi(i)} U_m$$

by our choice of λ. □

Putting together the previous two lemmas, we get:

Proposition 5. *If $h \in B_\gamma(g)$, then h has the property that every γ-pseudo-orbit is $\frac{1}{n}$-shadowed.*

Proof. Fix $h \in B_\gamma(g)$, and let $\langle x_j \rangle$ be a γ-pseudo-orbit for h. For each j, choose some $I(j) \in \{1,\dots,k\}$ such that $x_j \in U_{I(j)}$. Thus $I : \mathbb{N} \to \{1,\dots,k\}$ is a function describing the itinerary of our pseudo-orbit. By Lemma 3.7, $I(j+1) \in \phi(I(j))$ for every $j \in \mathbb{N}$; in other words, I describes a walk through Φ. By Lemma 3.6,

$$h\big(A_{I(j),I(j+1)}\big) \supseteq A_{I(j+1),I(j+2)}$$

for every $j \in \mathbb{N}$. From this and the compactness of D, we may conclude that

$$\bigcap_{j\in\mathbb{N}} h^{-j}\big(A_{I(j),I(j+1)}\big) \ne \emptyset.$$

Thus there is some $y \in A_{I(0),I(1)}$ such that

$$h^j(y) \in A_{I(j),I(j+1)} \subseteq U_{I(j)}$$

for every $j \in \mathbb{N}$. By the definition of I, $x_j \in U_{I(j)}$ for all $j \in \mathbb{N}$ as well. Thus

$$d(h^j(y), x_j) < \operatorname{diam}(U_{I(j)}) < \frac{\varepsilon}{2} < \frac{1}{n}$$

for every $j \in \mathbb{N}$. Hence every γ-pseudo-orbit for h is ε-shadowed. □

Corollary 1. *The set $\mathcal{R}_n$ of all $h \in \mathcal{C}(D)$ with the property that there is some $\gamma > 0$ such that every γ-pseudo-orbit for h is ε-shadowed has dense interior in (D).*

This corollary completes the proof of the theorem: we have showed that the set $\mathcal{R}_n$ described above has dense interior for arbitrary $n \in \mathbb{N}$. Thus $\mathcal{R} = \bigcap_{n\in\mathbb{N}} \mathcal{R}_n$ is co-meager in $\mathcal{C}(D)$. As $\mathcal{R}$ is precisely the set of functions in $\mathcal{C}(D)$ with shadowing, we are done.

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