CONVERGENCE OF SEQUENCES OF INVERSE LIMITS

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Abstract. We consider the following question. Let X be a compact metric space, and let $(f_n : X \to 2^X)_{n=1}^{\infty}$ be a sequence of upper semi-continuous set-valued functions whose graphs converge to the graph of a function $f : X \to 2^X$ in the hyperspace $2^{X\times X}$. Under what additional assumptions does it follow that the corresponding sequence of inverse limits $(\varprojlim \mathbf{f_n})_{n=1}^{\infty}$ converges to $\varprojlim \mathbf{f}$ in the hyperspace $2^{\prod_{i=1}^{\infty} X}$?

We give two nonequivalent conditions which generalize previous answers given by Banič, Črepnjak, Merhar, and Milutinović in 2010 and 2011.

Introduction

We will examine the question below which Banič, Črepnjak, Merhar, and Milutinović considered in [1, 2]. The following notation will enable the question to be stated more succinctly.

Let X, be a compact metric space, and let $\mathscr{X}$ represent the product space $\prod_{i=1}^{\infty} X$. Given an upper semi-continuous function $f : X \to 2^X$ and a sequence $(f_n : X \to 2^X)_{n=1}^{\infty}$ of upper semi-continuous functions, let $K = \varprojlim \mathbf{f}$ and $K_n = \varprojlim \mathbf{f_n}$ for each $n \in \mathbb{N}$. Additionally, let $\Gamma(f) = \{(x, y) \in X \times X : y \in f(x)\}$, and likewise for each function f_n.

Question. If $\lim_n \Gamma(f_n) = \Gamma(f)$ in the hyperspace $2^{X\times X}$, under what additional assumptions does it follow that $\lim_n K_n = K$ in the hyperspace $2^{\mathscr{X}}$?

Banič, Črepnjak, Merhar, and Milutinović gave a partial answer in [2, Theorem 3.3], stating that the statement $\lim_n K_n = K$ holds so long as f is continuous and single-valued (i.e. $f : X \to X$), and $\pi_1(K) \subseteq \liminf_n \pi_1(K_n)$. We demonstrate that the condition that f be continuous and single-valued may be relaxed in two ways yielding the following theorems.

Theorem 1. *Let X be a compact metric space and $f : X \to 2^X$ be upper semi-continuous. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be an upper semi-continuous function such that $\lim_n \Gamma(f_n) = \Gamma(f)$ in $2^{X\times X}$. If $\pi_1(K) \subseteq \liminf_n \pi_1(K_n)$ and K has the weak compact full projection property, then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.*

Theorem 2. *Let X be a compact metric space and $f : X \to 2^X$ be continuous. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be upper semi-continuous. If $\pi_1(K) \subseteq \liminf_n \pi_1(K_n)$, and there exists a set $A \subseteq X$ such that*

1. *A is dense in X,*
2. *for each $a \in A$, $A \cap f(a)$ is dense in $f(a)$,*
3. *$A \subseteq f(A)$, and*
4. *for each $a \in A$, $\lim_n f_n(a) = f(a)$ in 2^X,*

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then $\lim_n K_n = K$ *in* $2^{\mathscr{X}}$.

Theorem 1 is proven in Section 2, and Theorem 2 in Section 3. Additionally, it will be shown that both of these theorems are generalizations of the result of Banič et al. In Section 4, Theorems 1 and 2 are generalized further to be applicable to non-constant inverse sequences.

Finally, a few examples are given in Section 5.

1. Preliminaries

If X is a compact metric space, we denote by 2^X the set of all non-empty compact subsets of X. This space, 2^X, is referred to as the *hyperspace of* X. If X and Y are compact metric spaces and $x \in X$, a function $f : X \to 2^Y$ is said to be *upper semi-continuous* at x if for every open set $V \subseteq Y$ containing $f(x)$, there exists an open set $U \subseteq X$ containing x such that $f(t) \subseteq V$ for all $t \in U$. The function f is said to be *upper semi-continuous* if it is upper semi-continuous at each point of X. The *graph* of a function $f : X \to 2^Y$, denoted $\Gamma(f)$, is the subset of $X \times Y$ consisting of all points (x, y) such that $y \in f(x)$. In [6], it was shown that if X and Y are compact metric spaces, a function $f : X \to 2^Y$ is upper semi-continuous if and only if $\Gamma(f)$ is compact.

Suppose $\mathbf{X} = (X_i)_{i\in\mathbb{N}}$ is a sequence of compact metric spaces, and $\mathbf{f} = (f_i)_{i\in\mathbb{N}}$ is a sequence of upper semi-continuous functions such that for each $i \in \mathbb{N}$, $f_i : X_{i+1} \to 2^{X_i}$. Then the pair $\{\mathbf{X}, \mathbf{f}\}$ is called an *inverse sequence*, and the *inverse limit* of that inverse sequence, denoted $\varprojlim \mathbf{f}$, is the set

$$\varprojlim \mathbf{f} = \{\mathbf{x} \in \prod_{i=1}^{\infty} X_i : x_i \in f_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\}.$$

(In this paper, sequences–both finite and infinite–will be written in bold, and their terms will be written in italics.) The spaces, X_i, are called the *factor spaces* of the inverse sequence; and the upper semi-continuous functions, f_i, are called the *bonding functions* of the inverse sequence. Given any compact metric space X and an upper semi-continuous function $f : X \to 2^X$, there is a naturally induced inverse sequence $\{\mathbf{X}, \mathbf{f}\}$ where for each $i \in \mathbb{N}$, $X_i = X$ and $f_i = f$.

If $\mathbf{X}$ is a sequence of compact metric spaces and $j \in \mathbb{N}$, the *projection maps*

$$\pi_j : \prod_{i=1}^{\infty} X_i \to X_j \text{ and } \pi_{[1,j]} : \prod_{i=1}^{\infty} X_i \to \prod_{i=1}^{j} X_i$$

are defined by $\pi_j(\mathbf{x}) = x_j$, and $\pi_{[1,j]}(\mathbf{x}) = (x_1, \ldots, x_j)$. If $\{\mathbf{X}, \mathbf{f}\}$ is an inverse sequence, then we will typically consider these maps to have $\varprojlim \mathbf{f}$ as their domain rather than writing $\pi_j|_{\varprojlim \mathbf{f}}$ or $\pi_{[1,j]}|_{\varprojlim \mathbf{f}}$.

If $\{\mathbf{X}, \mathbf{f}\}$ is an inverse sequence we will suppose that the metric on each factor space is bounded by 1 and define a metric D on the inverse limit by

$$D(\mathbf{x}, \mathbf{y}) = \sum_{n=1}^{\infty} \frac{d_n(x_n, y_n)}{2^n}.$$

Central to this paper, is the notion of convergence of a sequence of closed sets. This convergence is viewed with respect to a metric on the hyperspace. We define this metric here.

Definition 1.1. Suppose X is a compact metric space with metric d. If $A \subseteq X$ is closed, and $\epsilon > 0$, then

$$N(A, \epsilon) = \{x \in X : d(x, a) < \epsilon \text{ for some } a \in A\}.$$

The *Hausdorff metric* $\mathcal{H}_d$ on 2^X is defined by

$$\mathcal{H}_d(A, B) = \inf\{\epsilon > 0 : A \subseteq N(B, \epsilon), \text{ and } B \subseteq N(A, \epsilon)\}.$$

Given a sequence $(A_n)_{n=1}^{\infty}$ of closed subsets of X, we define the limit of the sequence, $\lim_n A_n$, to be the limit with respect to the metric $\mathcal{H}_d$.

There is another equivalent way to view limits of sequences of closed sets.

Definition 1.2. Suppose X is a compact metric space, and $(A_n)_{n=1}^{\infty}$ is a sequence of closed subsets of X. Then

$$\begin{aligned} \limsup_n A_n &= \{x \in X : \text{ for all } \epsilon > 0, B(x, \epsilon) \cap A_n \neq \emptyset \text{ for infinitely many } n \in \mathbb{N}\}, \text{ and} \\ \liminf_n A_n &= \{x \in X : \text{ for all } \epsilon > 0, B(x, \epsilon) \cap A_n \neq \emptyset \text{ for all but finitely many } n \in \mathbb{N}\} \end{aligned}$$

(where $B(x, \epsilon)$ represents the ball of radius ϵ, centered at x).

A proof of the following theorem can be found in [8, p.57].

Theorem 1.3. *Let X be a compact metric space, and let $(A_n)_{n=1}^{\infty}$ be a sequence of closed subsets of X. Then $\lim_n A_n = A$ if and only if $A = \liminf_n A_n = \limsup_n A_n$.*

A *continuum* is a non-empty, compact, connected metric space. A continuum which is a subset of another continuum X is called a *subcontinuum of X*. The following theorem concerns sequences of continua. A proof may be found in [8, p.61].

Theorem 1.4. *Let X be a compact metric space, and let $(A_n)_{n=1}^{\infty}$ be a convergent sequence of closed subsets of X. If for each $n \in \mathbb{N}$, A_n is a continuum, then $\lim_n A_n$ is a continuum.*

Before we begin Section 2, we make a few additional observations and solidify some notation and terminology which will be utilized throughout the rest of this paper.

Notation. Given a compact metric space X, a sequence of closed subsets of X, $(X_n)_{n=1}^{\infty}$, an upper semi-continuous function $f : X \to 2^X$, and a sequence of upper semi-continuous functions $(f_n : X_n \to 2^{X_n})_{n=1}^{\infty}$, let $\mathscr{X} = \prod_{i=1}^{\infty} X$, $K = \varprojlim \mathbf{f}$, and for each $n \in \mathbb{N}$, $K_n = \varprojlim \mathbf{f_n}$.

An upper semi-continuous function $f : X \to 2^Y$ is called *surjective* if for every $y \in Y$, there exists $x \in X$ such that $y \in f(x)$. Given two upper semi continuous functions $f : X \to 2^Y$ and $g : Y \to 2^Z$, their composition $g \circ f$ is defined to be

$$g \circ f(x) = \{z \in Z : z \in g(y) \text{ for some } y \in f(x)\}.$$

Given a single upper semi-continuous function $f : X \to 2^X$, f^2 is defined to be $f \circ f$, and for $n \geq 3$, f^n is defined to be $f \circ f^{n-1}$.

Remark 1.5. Let X be a compact metric space and $f : X \to 2^X$ be upper semi-continuous. If $Y = \bigcap_{n=1}^{\infty} f^n(X)$, and $g : Y \to 2^Y$ is defined by $g = f|_Y$, then $\varprojlim \mathbf{f} = \varprojlim \mathbf{g}$.

Thus, when looking at inverse limits, it is natural to suppose that the bonding functions are surjective. In light of this, we rephrase the statement of [2, Theorem 3.3] in the following way.

Theorem 1.6 (Banič, Črepnjak, Merhar, Milutinović)**.** *Let X be a compact metric space, and $f : X \to X$ be a surjective continuous function. For each $n \in \mathbb{N}$, let X_n be a closed subset of X, and let $f_n : X_n \to 2^{X_n}$ be a surjective upper semi-continuous function such that* $\lim_n \Gamma(f_n) = \Gamma(f)$ *in* $2^{X \times X}$*. Then* $\lim_n K_n = K$ *in* $2^{\mathscr{X}}$ *if and only if* $\lim_n X_n = X$ *in* 2^X*.*

2. Weak Full Projection Property

We now begin our first generalization of Theorem 1.6. In this section, it is shown that the limit function $f : X \to 2^X$ need not be single-valued so long as its inverse limit K has the weak compact full projection property. In fact, if for each $n \in \mathbb{N}$, K_n is connected, then K need only have the weak continuum full projection property.

This is a generalization of Theorem 1.6, because the inverse limit of a continuous single-valued function has the compact full projection property. (This fact follows from [4, Theorem 1.9].) We begin by defining the various forms of the full projection property.

Definition 2.1. Let X be a compact metric space, and let $f : X \to 2^X$ be a surjective upper semi-continuous function. We say that $\varprojlim \mathbf{f}$ has the *compact (continuum) full projection property* if, given a compact set (continuum) $H \subseteq \varprojlim \mathbf{f}$ such that $\pi_i(H) = X$ for infinitely many $i \in \mathbb{N}$, it follows that $H = \varprojlim \mathbf{f}$.

We say that $\varprojlim \mathbf{f}$ has the *weak compact (continuum) full projection property* if, given a compact set (continuum) $H \subseteq \varprojlim \mathbf{f}$ such that $\pi_i(H) = X$ for all $i \in \mathbb{N}$, it follows that $H = \varprojlim \mathbf{f}$.

The (weak) compact (continuum) full projection property is one of many important properties of inverse limits with single-valued bonding maps which does not hold in the setting of set-valued maps. For discussion of this property, see [3, 5, 7] and others.

The following theorem is found in [1, Theorem 3.2].

Theorem 2.2 (Banič, Črepnjak, Merhar, Milutinović)**.** *Let X be a compact metric space and $f : X \to 2^X$ be an upper semi-continuous function. For each positive integer n, let $f_n : X \to 2^X$ be an upper semi-continuous function such that* $\lim_n \Gamma(f_n) = \Gamma(f)$ *in* $2^{X \times X}$*. If $P \subseteq \mathscr{X}$ is an accumulation point of the sequence $(K_n)_{n=1}^{\infty}$ in the hyperspace $2^{\mathscr{X}}$, then $P \subseteq K$.*

In light of Remark 1.5, we may reword the statement of Theorem 2.2 as follows.

Lemma 2.3. *Let X be a compact metric space and $f : X \to 2^X$ be a surjective upper semi-continuous function. For each $n \in \mathbb{N}$, let X_n be a closed subset of X, and $f_n : X_n \to 2^{X_n}$ be a surjective upper semi-continuous function such that* $\lim_n \Gamma(f_n) = \Gamma(f)$ *in* $2^{X \times X}$*. If $P \subseteq \mathscr{X}$ is an accumulation point of the sequence $(K_n)_{n=1}^{\infty}$ in the hyperspace $2^{\mathscr{X}}$, then $P \subseteq K$.*

The next theorem demonstrates that accumulation points have full projection in every coordinate.

Lemma 2.4. *Let X be a compact metric space. For each $n \in \mathbb{N}$, let X_n be a closed subset of X, and $f_n : X_n \to 2^{X_n}$ be a surjective upper semi-continuous function such that $\lim_n X_n = X$ in 2^X. If $P \subseteq \mathscr{X}$ is an accumulation point of the sequence $(K_n)_{n=1}^{\infty}$ in the hyperspace $2^{\mathscr{X}}$, then $\pi_i(P) = X$ for all $i \in \mathbb{N}$.*

Proof. Let $P \subseteq \mathscr{X}$ be an accumulation point of the sequence $(K_n)_{n=1}^{\infty}$, and let $(K_{n_j})_{j=1}^{\infty}$ be a subsequence which converges to P. Fix $i \in \mathbb{N}$. To show that $X \subseteq \pi_i(P)$, let $y \in X$. Since $\lim_j X_{n_j} = \lim_n X_n = X$, we have that $y \in \lim_j X_{n_j}$, so there exists a sequence $(y_j)_{j=1}^{\infty}$ whose limit is y where for each $j \in \mathbb{N}$, $y_j \in X_{n_j}$.

For each $j \in \mathbb{N}$, the function $f_{n_j} : X_{n_j} \to 2^{X^{n_j}}$ is surjective, so there exists a point $\mathbf{x}_j \in K_{n_j}$ with $\pi_i(\mathbf{x}_j) = y_j$. The resulting sequence $(\mathbf{x}_j)_{j=1}^{\infty}$ has a convergent subsequence $(\mathbf{x}_{j_k})_{k=1}^{\infty}$ converging to a point $\mathbf{x}$. Since the sequence $(K_{n_j})_{j=1}^{\infty}$ converges to P, it follows that its subsequence $(K_{n_{j_k}})_{k=1}^{\infty}$ converges to P as well. Thus, since $\mathbf{x}$ is the limit of the sequence $(\mathbf{x}_{j_k})_{k=1}^{\infty}$ where for each $k \in \mathbb{N}$, $\mathbf{x}_{j_k} \in K_{n_{j_k}}$, we have that $\mathbf{x} \in P$.

Moreover,

$$\pi_i(\mathbf{x}) = \lim_{k\to\infty} \pi_i(\mathbf{x}_{j_k}) = \lim_{k\to\infty} y_{j_k} = \lim_{j\to\infty} y_j = y.$$

Therefore, $y \in \pi_i(P)$. Since this holds for all $y \in X$ and $i \in \mathbb{N}$, it follows that $X \subseteq \pi_i(P)$ for all $i \in \mathbb{N}$. □

This brings us to our primary result for the section.

Theorem 2.5. *Let X be a compact metric space and $f : X \to 2^X$ be a surjective upper semi-continuous function. For each $n \in \mathbb{N}$, let X_n be a closed subset of X, and $f_n : X_n \to 2^{X_n}$ be a surjective upper semi-continuous function such that $\lim_n \Gamma(f_n) = \Gamma(f)$ in $2^{X\times X}$. If K has the weak compact full projection property, then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.*

Proof. First, note that since each f_n is surjective and $\lim_n \Gamma(f_n) = \Gamma(f)$, it follows that $\lim_n X_n = X$. Next, note that since $2^{\mathscr{X}}$ is a compact space $(K_n)_{n=1}^{\infty}$ has an accumulation point $P \subseteq \mathscr{X}$.

By Lemma 2.3, P is a subset of K, and by Lemma 2.4, $\pi_i(P) = X$ for all $i \in \mathbb{N}$. Therefore, since K has the weak compact full projection property, we have that $P = K$.

This means that K is the only accumulation point of $(K_n)_{n=1}^{\infty}$, and therefore, $\lim_n K_n = K$. □

The following corollary follows immediately given Theorem 1.4.

Corollary 2.6. *Let X be a compact metric space and $f : X \to 2^X$ be a surjective upper semi-continuous function. For each $n \in \mathbb{N}$, let X_n be a closed subset of X, and $f_n : X_n \to 2^{X_n}$ be a surjective upper semi-continuous function such that $\lim_n \Gamma(f_n) = \Gamma(f)$ in $2^{X\times X}$. If for each $n \in \mathbb{N}$, K_n is a continuum, and K has the weak continuum full projection property, then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.*

3. Continuous Limit Function and Pointwise Convergence

This section is concerned with a second generalization of Theorem 1.6. Here, we demonstrate that the limit function $f : X \to 2^X$ need not be single-valued so long as it satisfies two other conditions. First, f must be continuous with respect to the Hausdorff metric on 2^X. Second, there must exist a set A, meeting the criteria of Lemma 3.2 below, such that for every point a of A, a is in X_n for all but finitely many $n \in \mathbb{N}$, and $\lim_n f_n(a) = f(a)$ in 2^X.

The following Lemma ([1, Lemma 3.3]) is critical in the proof of Theorem 1.6.

Lemma 3.1 (Banič, Črepnjak, Merhar, Milutinović). *Let X be a compact metric space, and let $f : X \to X$ be a continuous single-valued function. For each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be an upper semi-continuous function such that $\lim_n \Gamma(f_n) = \Gamma(f)$ in $2^{X\times X}$. Then for each $\epsilon > 0$, there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, for all $x \in X$, and for all $y \in f_n(x)$, $d(y, f(x)) < \epsilon$.*

The importance of this lemma is the conclusion that for all $x \in X$, $\lim_n f_n(x) = f(x)$ in 2^X. As we show in Theorem 3.3, pointwise convergence on a sufficiently nice dense set is sufficient, provided $f : X \to 2^X$ is continuous.

Lemma 3.2. *Let X be a compact metric space, and $f : X \to 2^X$ be continuous. Suppose there exists a set A which is dense in X such that $A \subseteq f(A)$, and for each $a \in A$, $A \cap f(a)$ is dense in $f(a)$. Then the set*

$$\mathcal{A} = \{\mathbf{x} \in \varprojlim \mathbf{f} : x_i \in A \text{ for all } i \in \mathbb{N}\}$$

is dense in $\varprojlim \mathbf{f}$.

Proof. Let $\mathbf{x} \in \varprojlim \mathbf{f}$, and let $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that

$$\sum_{i=m+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2}.$$

Let $\delta_1 = \epsilon/2$, and choose a positive number $\delta_2 < \epsilon/2$ such that if $a, b \in X$ with $d(a, b) < \delta_2$, then $\mathcal{H}_d(f(a), f(b)) < \delta_1$.

Suppose that for some $k \leq m-1$, δ_i has been defined for all $1 \leq i \leq k$. Then choose a positive number $\delta_{k+1} < \epsilon/2$ so that if $a, b \in X$ with $d(a, b) < \delta_{k+1}$, then $\mathcal{H}_d(f(a), f(b)) < \delta_k$. In this way, we define a finite sequence $(\delta_i)_{i=1}^m$.

We now begin the construction of an element $\mathbf{a} \in \mathcal{A}$ with $D(\mathbf{x}, \mathbf{a}) < \epsilon$.

Since A is dense in X, there exists $a_m \in A$ with $d(a_m, x_m) < \delta_m$. Then $\mathcal{H}_d(f(a_m), f(x_m)) < \delta_{m-1}$. This means that the open ball of radius δ_{m-1}, centered at x_{m-1} intersects $f(a_m)$, and since $A \cap f(a_m)$ is dense in $f(a_m)$, there is a point $a_{m-1} \in A \cap f(a_m)$ such that $d(a_{m-1}, x_{m-1}) < \delta_{m-1}$. Similarly, then $\mathcal{H}_d(f(a_{m-1}), f(x_{m-1})) < \delta_{m-2}$, and it follows from the density of $A \cap f(a_{m-1})$ in $f(a_{m-1})$ that there exists a point $a_{m-2} \in A \cap f(a_{m-1})$ such that $d(a_{m-2}, x_{m-2}) < \delta_{m-2}$.

We may continue in this manner until we have chosen $a_1, a_2, \ldots, a_m$ such that for all $i = 1, \ldots, m$, $d(a_i, x_i) < \delta_i \leq \epsilon/2$. Then, since $A \subseteq f(A)$, for each $i \geq m$, we may choose $a_{i+1} \in A \cap f^{-1}(a_i)$. Let $\mathbf{a} = (a_i)_{i=1}^{\infty}$. Then $\mathbf{a} \in \mathcal{A}$. Moreover,

$$D(\mathbf{x}, \mathbf{a}) = \sum_{i=1}^{\infty} \frac{d(x_i, a_i)}{2^i} = \sum_{i=1}^{m} \frac{d(x_i, a_i)}{2^i} + \sum_{i=m+1}^{\infty} \frac{d(x_i, a_i)}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

□

This brings us to the main result of this section.

Theorem 3.3. *Let X be a compact metric space and $f : X \to 2^X$ be surjective and continuous. For each $n \in \mathbb{N}$, let X_n be a closed subset of X and $f_n : X_n \to 2^{X_n}$ be a surjective upper semi-continuous function. If there exists a set A such that*

(1) *A meets the criteria of Lemma 3.2 with respect to $f : X \to 2^X$, and*

(2) *for each $a \in A$, there exists an $N \in \mathbb{N}$ such that $a \in X_n$ for all $n \geq N$, and the sequence $(f_n(a))_{n=N}^\infty$ converges to $f(a)$ in 2^X,*

then $\lim_n K_n = K$ *in* $2^{\mathscr{X}}$.

Proof. We will prove this by showing that $\limsup_n K_n \subseteq K \subseteq \liminf_n K_n$. From Lemma 2.3, every accumulation point of $(K_n)_{n=1}^\infty$ is a subset of K. Thus, since

$$\limsup_n K_n = \cup\{P : P \text{ is an accumulation point of } (K_n)_{n=1}^\infty\},$$

we have that $\limsup_n K_n \subseteq K$.

Now, since $\liminf_n K_n$ is closed, to show that $K \subseteq \liminf_n K_n$, it will suffice to show that

$$\mathcal{A} = \{\mathbf{x} \in K : x_i \in A \text{ for all } i \in \mathbb{N}\} \subseteq \liminf_n K_n.$$

Towards this end, let $\mathbf{x} \in \mathcal{A}$. We will show that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$, there is a point $\mathbf{y} \in K_n$ which is within ϵ of $\mathbf{x}$.

Let $\epsilon > 0$. First, choose $m \in \mathbb{N}$, such that

$$\sum_{i=m+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2},$$

Next, let $\delta_{m-1} = \epsilon/2$, and choose any positive number ϵ_{m-2} less than δ_{m-1}. Since $f : X \to 2^X$ is continuous, we may choose a positive number δ_{m-2} less than ϵ_{m-2} so that if $a, b \in X$ with $d(a,b) < \delta_{m-2}$, then $\mathcal{H}_d(f(a), f(b)) < \epsilon_{m-2}$. Then choose ϵ_{m-3} to be any positive number less than δ_{m-2}, and choose a positive number δ_{m-3} less than ϵ_{m-3} so that if $a, b \in X$ with $d(a,b) < \delta_{m-3}$, then $\mathcal{H}_d(f(a), f(b)) < \epsilon_{m-3}$.

We continue on in this manner until we choose a positive number ϵ_1 less than δ_2, and we choose a positive number δ_1 less than ϵ_1 such that if $a, b \in X$ with $d(a,b) < \delta_1$, then $\mathcal{H}_d(f(a), f(b)) < \epsilon_1$. This results in a sequence

$$0 < \delta_1 < \epsilon_1 < \delta_2 < \epsilon_2 < \cdots < \delta_{m-2} < \epsilon_{m-2} < \delta_{m-1} = \frac{\epsilon}{2}.$$

Now, let

$$\epsilon_0 = \min\{\delta_1, \delta_{i+1} - \epsilon_i : i = 1, \ldots, m-2\}.$$

From Property (2) above, we have that since $x_1, \ldots, x_m \in A$, there exists $M \in \mathbb{N}$ such that $x_1, \ldots, x_m \in X_n$ for all $n \geq M$, and for each $i = 1, \ldots, m-1$, the sequence $(f_n(x_{i+1}))_{n=M}^\infty$ converges to $f(x_{i+1})$ in 2^X. Hence, we may choose a natural number $N \geq M$ so that for each $n \geq N$ and each $i = 1, \ldots, m-1$, we have that $\mathcal{H}_d(f_n(x_{i+1}), f(x_{i+1})) < \epsilon_0$.

Fix $n \geq N$, and let $y_m = x_m$. Since $x_{m-1} \in f(x_m)$, by our choice of N, there exists an element $y_{m-1} \in f_n(x_m)$ such that $d(y_{m-1}, x_{m-1}) < \epsilon_0 \leq \delta_1 < \epsilon/2$.

Suppose that for some natural number $k \leq m-2$, an element y_{m-i} has been chosen for each $i = 1, \ldots, k$ such that $y_{m-i} \in f_n(y_{m-i+1})$, and $d(x_{m-i}, y_{m-i}) < \delta_i$. Then from the choice of δ_k, we have that

$\mathcal{H}_d(f(x_{m-k}), f(y_{m-k})) < \epsilon_k$. Additionally, by our choice of N, we have that $\mathcal{H}_d(f(y_{m-k}), f_n(y_{m-k})) < \epsilon_0$, so it follows that

$$\begin{aligned} & \mathcal{H}_d\left(f(x_{m-k}), f_n(y_{m-k})\right) \\ \leq \quad & \mathcal{H}_d\left(f(x_{m-k}), f(y_{m-k})\right) + \mathcal{H}_d\left(f(y_{m-k}), f_n(y_{m-k})\right) \\ < \quad & \epsilon_k + \epsilon_0 \\ \leq \quad & \delta_{k+1}. \end{aligned}$$

Therefore, there exists $y_{m-k-1} \in f_n(y_{m-k})$ such that $d(x_{m-k-1}, y_{m-k-1}) < \delta_{k+1} \leq \epsilon/2$. In this manner, for each $i = 1, \ldots, m-1$ a point y_i is chosen so that $d(x_i, y_i) < \epsilon/2$.

Since $f_n : X_n \to 2^{X_n}$ is surjective, for each $i \geq m$, we may choose $y_{i+1} \in f_n^{-1}(y_i)$. Thus, y_i has been defined for all $i \in \mathbb{N}$, so let $\mathbf{y} = (y_i)_{i=1}^{\infty}$. Then

$$D(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{d(x_i, y_i)}{2^i} = \sum_{i=1}^{m} \frac{d(x_i, y_i)}{2^i} + \sum_{i=m+1}^{\infty} \frac{d(x_i, y_i)}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Therefore, $\mathbf{x} \in \liminf_n K_n$, and we have that $\limsup_n K_n \subseteq K \subseteq \liminf_n K_n$. Thus, $\lim_n K_n = K$. □

4. Non-constant Inverse Sequences

Both Theorem 2.5 and Theorem 3.3 may be stated more generally for inverse sequences that are not induced by a single function on a compact metric space. These statements are given below. Their proofs are, in essence, identical to the proofs of the previous two sections.

For the following theorems, for a given $n \in \mathbb{N}$, the terms of a sequence $\mathbf{X_n}$ will be denoted $X_1^n, X_2^n, \ldots$. Likewise, the terms of a sequence $\mathbf{f_n}$ will be denoted $f_1^n, f_2^n, \ldots$.

Theorem 4.1. *Let $\{\mathbf{X}, \mathbf{f}\}$ be an inverse sequence with surjective bonding functions, and for each $n \in \mathbb{N}$, let $\{\mathbf{X_n}, \mathbf{f_n}\}$ be an inverse sequence with surjective bonding functions, such that for each $i \in \mathbb{N}$, $X_i^n \subseteq X_i$. If for each $i \in \mathbb{N}$, $\lim_n \Gamma(f_i^n) = \Gamma(f_i)$ in $2^{X_{i+1} \times X_i}$, and K has the weak compact full projection property, then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.*

Corollary 4.2. *Let $\{\mathbf{X}, \mathbf{f}\}$ be an inverse sequence with surjective bonding functions, and for each $n \in \mathbb{N}$, let $\{\mathbf{X_n}, \mathbf{f_n}\}$ be an inverse sequence with surjective bonding functions, such that for each $i \in \mathbb{N}$, $X_i^n \subseteq X_i$. Suppose that for each $n \in \mathbb{N}$, K_n is a continuum. If for each $i \in \mathbb{N}$, $\lim_n \Gamma(f_i^n) = \Gamma(f_i)$ in $2^{X_{i+1} \times X_i}$, and K has the weak continuum full projection property, then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.*

Theorem 4.3. *Let $\{\mathbf{X}, \mathbf{f}\}$ be an inverse sequence such that each bonding function is continuous. For each $n \in \mathbb{N}$, let $\{\mathbf{X_n}, \mathbf{f_n}\}$ be an inverse sequence such that for all $i \in \mathbb{N}$, $X_i^n \subseteq X_i$, and f_i^n is surjective. If there exists a sequence $(A_i)_{i=2}^{\infty}$ such that for each $i \in \mathbb{N}$,*

(1) *A_i is dense in X_i,*
(2) *for each $a \in A_{i+1}$, $A_i \cap f_i(a)$ is dense in $f_i(a)$,*
(3) *$A_i \subseteq f_i(A_{i+1})$, and*
(4) *for each $a \in A_{i+1}$, a is in X_{i+1}^n for all but finitely many $n \in \mathbb{N}$, and $\lim_n f_i^n(a) = f_i(a)$ in 2^{X_i},*

then $\lim_n K_n = K$.

5. Examples

The following examples illustrate applications of the main results of this paper. In particular, we show that the each is a nontrivial generalization of Theorem 1.6. Our first example appears in [1, Example 4.5]. We will show that, while the main result of that paper (Theorem 1.6) does not apply, Theorem 1 does.

The following definition will be useful.

Definition 5.1. Let X be a compact metric space, and let $f : X \to 2^X$ be upper semi-continuous. Then for all $n \in \mathbb{N}$, define

$$\Gamma'_n = \{\mathbf{x} \in \prod_{i=1}^{n} [0,1] : x_i \in f(x_{i+1}) \text{ for all } 1 \leq i < n\}.$$

Example 5.2. For each $n \in \mathbb{N}$, let $f : [0,1] \to 2^{[0,1]}$ be given by $f(x) = x^n$, and let $f : [0,1] \to 2^{[0,1]}$ be the function given by $f(x) = 0$ for all $x \neq 1$, and $f(1) = [0,1]$ (pictured in Figure 1). Then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.

Proof. To apply Theorem 1, we need only show that $\varprojlim \mathbf{f}$ has the weak compact full projection property. To do so, it suffices to show that for any $n \in \mathbb{N}$, if H is a compact subset of Γ'_n with $p_i(H) = [0,1]$ for all $i = 1, \dots, n$, then $H = \Gamma'_n$ (where $p_i : \Gamma'_n \to [0,1]$ is projection onto the ith coordinate).

Fix $n \in \mathbb{N}$, and suppose that H is a compact subset of Γ'_n with $p_i(H) = [0,1]$ for all $i = 1, \dots, n$. For each natural number $k = 2, \dots, n-1$, define a set

$$A_k = \{\mathbf{x} \in \Gamma'_n : x_i = 0 \text{ for all } 1 \leq i \leq k-1, \text{ and } x_i = 1 \text{ for all } k+1 \leq i \leq n\}.$$

Additionally, define

$$\begin{aligned} A_1 &= \{\mathbf{x} \in \Gamma'_n : x_i = 1 \text{ for all } 2 \leq i \leq n\}, \text{ and} \\ A_n &= \{\mathbf{x} \in \Gamma'_n : x_i = 0 \text{ for all } 1 \leq i \leq n-1\}. \end{aligned}$$

Note that for each $k = 1, \dots, n$, A_k is an arc, and that $\Gamma'_n = \bigcup_{i=1}^{n} A_i$. We will show that for all $k = 1, \dots, n$, $A_k \subseteq H$. Let $\mathbf{x}$ be a point of A_k with $x_k \in (0,1)$. Notice that for any $j = 1, \dots, n$ with $j \neq k$, there are no points of A_j whose kth coordinate is x_k. Thus, since $x_k \in (0,1) \subseteq p_k(H)$, it follows that $\mathbf{x} \in H$. Then since H is compact, it also follows that $A_k \subseteq H$, and hence $\Gamma'_n \subseteq H$. $\square$

The next example is an application of Theorem 2, and provides an illustration that (even for continuous set-valued functions) convergence of $\Gamma(f_n)$ to $\Gamma(f)$ does not imply pointwise convergence of $f_n(x)$ to $f(x)$, and that pointwise convergence is not necessary.

Example 5.3. Let $g : [0,1] \to 2^{[0,1]}$ be the continuous set-valued function whose graph is the closed region of $[0,1] \times [0,1]$ bounded by four line segments: the first from $(0,1/2)$ to $(1/2,1)$, the second from $(1/2,1)$ to $(1,1/2)$, the third from $(1,1/2)$ to $(1/2,0)$, and the fourth from $(1/2,0)$ to $(0,1/2)$. The graph of g is pictured in Figure 3.

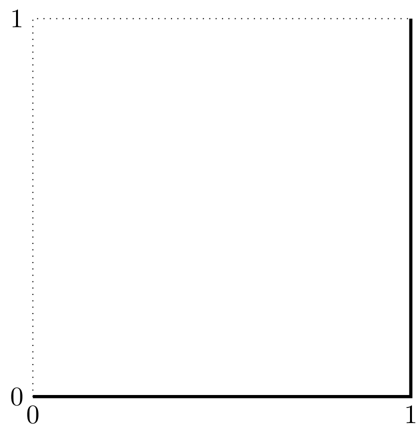

FIGURE 1. f

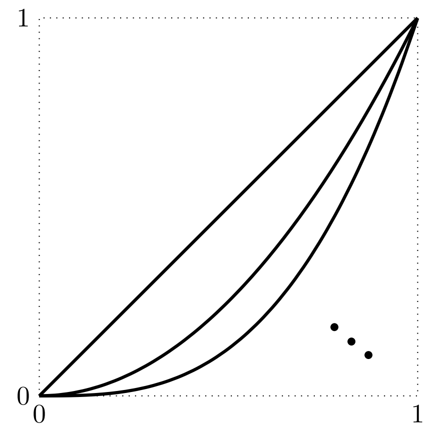

FIGURE 2. $(f_n)_{n=1}^{\infty}$

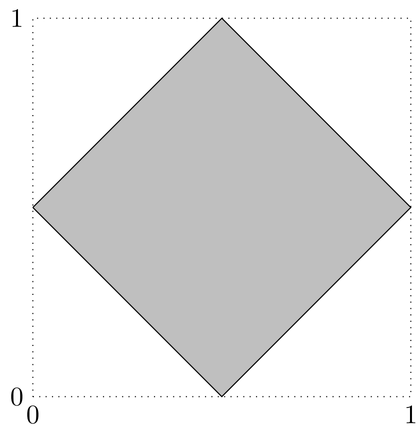

FIGURE 3. g

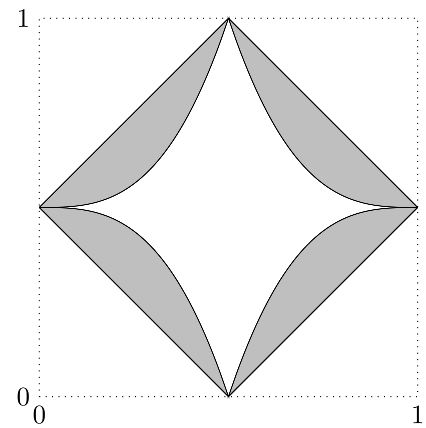

FIGURE 4. g_3

For each $n \in \mathbb{N}$, let $g_n : [0,1] \to 2^{[0,1]}$ be the upper semi-continuous function whose graph is equal to $\Gamma(g) \setminus L$ where L is the region of $[0,1] \times [0,1]$ bounded by the graphs of the following four functions:

$$\begin{aligned}
\varphi_1 &: [0,1/2] \to [1/2,1], & \varphi_1(x) &= 2^{n-1}x^n + \frac{1}{2}, \\
\varphi_2 &: [1/2,1] \to [1/2,1], & \varphi_2(x) &= 2^{n-1}(1-x)^n + \frac{1}{2}, \\
\varphi_3 &: [0,1/2] \to [0,1/2], & \varphi_3(x) &= -2^{n-1}x^n + \frac{1}{2}, \\
\varphi_4 &: [1/2,1] \to [0,1/2], & \varphi_4(x) &= -2^{n-1}(1-x)^n + \frac{1}{2},
\end{aligned}$$

The graph of g_3 is pictured in Figure 4.

Then if $K = \varprojlim \mathbf{g}$ and for each $n \in \mathbb{N}$, $K_n = \varprojlim \mathbf{g_n}$., then $\lim_n K_n = K$ in $2^{\mathscr{X}}$.

Proof. Let $A = [0,1] \setminus \{0, 1/2, 1\}$. Then A is dense in $[0,1]$, $g(A) = A$, and for all $a \in A$, $\lim_n g_n(a) = g(a)$ in $2^{[0,1]}$. Hence, by Theorem 3.3, $\lim_n K_n = K$ in $2^{\mathscr{X}}$. $\square$

The existence of a set A as in Theorem 3.3 is easy to identify in this example, but in general this may not be the case. However, it may be the case that such a set is not necessary for the conclusion to hold, leading to the following question.

Question 5.4. Let $f : X \to 2^X$ be a function, and for each $n \in \mathbb{N}$, let $f_n : X \to 2^X$ be upper semi-continuous such that $\lim_n \Gamma(f_n) = \Gamma(f)$, and $\pi_1(K) \subseteq \liminf_n \pi_1(K_n)$. If f is continuous, does it follow that $\lim_n K_n = K$ in $2^{\mathscr{X}}$?

The authors at one point speculated that the existence of a set A as in Theorem 3.3 would follow from the continuity of f, thus providing a positive answer to Question 5.4. We conclude with an example that illustrates that such a set A need not exist even if the limit function is continuous. This example does not, however, provide a negative answer to Question 5.4.

Example 5.5. Let $f : [0,1] \to 2^{[0,1]}$ be given by $f(x) = [0,1]$ for all $x \in [0,1]$. Let $(D_n)_{n=1}^{\infty}$ be a sequence of mutually disjoint subsets of $[0,1]$ such that for each $n \in \mathbb{N}$, the collection of open balls $\{B(x, 1/2^n) : x \in D_n\}$ is an open cover of $[0,1]$. For each $n \in \mathbb{N}$, define $f_n : [0,1] \to 2^{[0,1]}$ by

$$f_n(x) = \begin{cases} 0 & \text{for } x \notin D_n, \\ [0,1] & \text{for } x \in D_n. \end{cases}$$

Then f is continuous, $\lim_n \Gamma(f_n) = \Gamma(f)$ in $2^{[0,1]\times[0,1]}$, and $\lim_n K_n = K$ in $2^{\mathscr{X}}$, but for all $x \in [0,1]$, $\lim_n f_n(x) = \{0\}$ which is not equal to $f(x)$.

Proof. First, the fact that for all $x \in [0,1]$, $\lim_n f_n(x) = \{0\}$ in $2^{[0,1]}$, follows from the definition of the sequence $(f_n)_{n=1}^{\infty}$ and the fact that the members of the sequence $(D_n)_{n=1}^{\infty}$ are mutually disjoint. This means that for any $x \in [0,1]$, $f_n(x) = \{0\}$ for all but possibly one $n \in \mathbb{N}$.

To see that $\lim_n K_n = K$, note that by Lemma 2.3, $\limsup_n K_n \subseteq K$, so we must only show that $K \subseteq \liminf_n K_n$. Let $\mathbf{x} \in K$, and let $\epsilon > 0$. Choose $m \in \mathbb{N}$ such that

$$\sum_{i=m+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2},$$

and choose $N \in \mathbb{N}$ such that $2^{-N} < \epsilon/2$.

Fix a natural number $n \geq N$. By the definition of D_n, we have that there exists $y_m \in D_n$ such that $|x_m - y_m| < 2^{-n}$. Suppose that for some $k \leq m$, $y_k \in D_n$ has been defined. Then since $f_n(y_k) = [0,1]$, we may choose $y_{k-1} \in D_n$ such that $|x_{k-1} - y_{k-1}| < 2^{-n}$. In this way, we define $y_1, y_2, \ldots, y_m$ such that for all $i = 1, \ldots, m$, $|x_m - y_m| < 2^{-n} < \epsilon/2$.

Since f_n is surjective, for each $i \geq m$, we may choose $y_{i+1} \in f_n^{-1}(y_i)$. In this way we define a point $\mathbf{y} = (y_i)_{i=1}^{\infty} \in K_n$. Moreover, by construction,

$$D(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i} \leq \sum_{i=1}^{m} \frac{|x_i - y_i|}{2^i} + \sum_{i=m+1}^{\infty} \frac{1}{2^i} < \frac{\epsilon}{2} + \frac{\epsilon}{2}.$$

Therefore $\limsup_n K_n \subseteq K \subseteq \liminf_n K_n$, and hence $\lim_n K_n = K$.

∎

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