THE STRUCTURE OF LIMIT SETS FOR $\mathbb{Z}^d$ ACTIONS

JONATHAN MEDDAUGH AND BRIAN E. RAINES

Abstract. Central to the study of $\mathbb{Z}$ actions on compact metric spaces is the ω-limit set, the set of all limit points of a forward orbit. A closed set K is *internally chain transitive* provided for every $x, y \in K$ there is an ϵ-pseudo-orbit of points from K that starts with x and ends with y. It is known in several settings that the property of internal chain transitivity characterizes ω-limit sets. In this paper, we consider actions of $\mathbb{Z}^d$ on compact metric spaces. We give a general definition for shadowing and limit sets in this setting. We characterize limit sets in terms of a more general internal property which we call *internal mesh transitivity*.

1. Introduction

The study of $\mathbb{Z}^d$ actions on compact metric spaces is a natural generalization of the study of $\mathbb{Z}$ actions. Perhaps unsurprisingly many of the basic questions and results from the topological theory of $\mathbb{Z}$ actions are difficult to resolve for $\mathbb{Z}^d$ actions when $d \geq 2$. Consider for example shift spaces. Specifically, consider a countable product of a finite set of symbols, $\Sigma = \{0, 1, \ldots n\}$, with the discrete topology. Then the natural $\mathbb{Z}$ action on $\Sigma^{\mathbb{Z}}$, the shift map $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{Z}$, has many closed invariant subsets, *subshifts*. When the subshift consists of those patterns not containing any pattern from a finite collection of forbidden finite patterns we call it a *subshift of finite type (SFT)*. It is straightforward to compute the topological entropy of any SFT, and the technique has been known for some time, [13] and [9]. This cannot be said about subshifts of multidimensional shift spaces, $\Sigma^{\mathbb{Z}^d}$ $d \geq 2$,[7] and [8].

In this paper, we consider limit sets under $\mathbb{Z}^d$ actions for $d \geq 2$. For a $\mathbb{Z}$ action on a compact metric space, X, the *ω-limit set* of $x \in X$ is the set of limit points of the positive (forward) orbit of x. The *α-limit set* of $x \in X$ is the set of limit points of the negative (backward) orbit of x. In the case of $\mathbb{Z}^d$, though, there are no clear and agreed upon notions of ω and α-limit set.

Let $\Sigma = \{0, 1, \ldots, n\}$ and consider the one-dimensional $\sigma : \Sigma^{\mathbb{Z}} \to \Sigma^{\mathbb{Z}}$. Let $x = \ldots x_{-1}.x_0 x_1 \ldots$. Then it is easy to see that $y \in \omega(x)$ if, and only if, for every $n \in \mathbb{N}$ each central segment of $y = y_{-n} \ldots y_{-1}.y_0 \ldots y_n$ of length $2n + 1$ occurs infinitely often in $x_0 x_1 \ldots$. Similarly $y \in \alpha(x)$ if, and only if, for every $n \in \mathbb{N}$ each central segment of y of length $2n + 1$ occurs infinitely often in $\ldots x_{-2} x_{-1}$. In this spirit we can define limit sets for $\Sigma^{\mathbb{Z}^d}$, but we have many more directions to consider.

The first generalization along these lines was done by Oprocha, [11]. He defined some limit sets for $\mathbb{Z}^d$ shift spaces. In the two-dimensional setting, $\sigma : \Sigma^{\mathbb{Z}^2} \to$

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$\Sigma^{\mathbb{Z}^2}$, Oprocha defined four limit sets by considering the action restricted to each quadrant. Summarizing his work, let $x, y \in \Sigma^{\mathbb{Z}^2}$, then y is in the limit set of x generated by the first quadrant provided for every $n \in \mathbb{N}$ each central block of y of radius n occurs infinitely often in quadrant I of x centered at some sequence (a_i, b_i) for $i \in \mathbb{N}$ with $a_i \to \infty$ and $b_i \to \infty$. One way to visualize this condition is to imagine an infinite collection of L-shaped bands, $(L_j)_{j \in \mathbb{N}}$, with corner points at (j, j) on the integer lattice $\mathbb{Z}^2$. Then y is in x's first quadrant limit set if, and only if, each central block of y of radius n occurs above each L_j in x. He defines the three other limit sets for $\Sigma^{\mathbb{Z}^2}$ (and limit sets for $\Sigma^{\mathbb{Z}^d}$ for $d > 2$) analogously.

Our goal in this paper is to give a more general notion of limit set for $\mathbb{Z}^d$ actions and to give a characterization of these limit sets in terms of internal properties.

We extend Oprocha's notion to all $\mathbb{Z}^d$ actions and to any finite set of directions. This gives us many more types of limit set and one supra-limit set which we call the ω-limit set. The precise definition is given in the following sections, but in the case of a $\mathbb{Z}^2$ shift space it is easy to give a description of the ω-limit set. For each $j \in \mathbb{N}$ let M_j be the central block of $\mathbb{Z}^2$ of radius j. Let $x \in \Sigma^{\mathbb{Z}^2}$. Then we say $y \in \omega(x)$ if, and only if, for every $n \in \mathbb{N}$, each central block of y of radius n occurs in x outside of M_j. This is equivalent to saying there is some $(a, b) \in \mathbb{Z}^2$ with $|a| > j$ or $|b| > j$ such that $\sigma^{(a,b)}(x)$ and y agree on their central block of radius n. We prove that for a $\mathbb{Z}^d$ action, this ω-limit set contains all possible other limit sets as invariant subsets. It contains all limit points of the full orbit of x.

We also generalize the notion of shadowing to this setting and define an extension of internal chain transitivity which we call *internal mesh transitivity*. In $\mathbb{Z}$ actions, all ω-limit sets are internally chain transitive, but it is not always the case that all internally chain transitive sets are ω-limit sets. In the case of one-dimensional shift spaces and SFTs the two notions are equivalent, but there are one-dimensional sofic shift spaces with internally chain transitive sets which are not ω-limit sets, [2]. In the case of $\mathbb{Z}^d$ actions for $d \geq 2$, we prove that in many settings, such as d-dimensional shift spaces, internal mesh transitivity characterizes these generalized limit sets.

The paper is organized as follows. In Section 2, we give preliminary definitions and results for $\mathbb{Z}^d$ actions on compact metric spaces. We define the various limit sets including the ω-limit set, and we prove some of their basic topological properties. In Section 3, we generalize internal chain transitivity to $\mathbb{Z}^d$ actions and call it internal mesh transitivity. We prove that limit sets always have internal mesh transitivity. We show that in the presence of a strong type of shadowing (limit shadowing cf. [4]) internal mesh transitivity characterizes limit sets. In the last section, we turn our attention specifically to multidimensional shift spaces. We prove that these spaces have shadowing and limit shadowing, and therefore internal mesh transitivity characterizes limit sets. Finally, we give some conditions under which multidimensional shifts of finite type have the property that internal mesh transitivity characterizes limit sets.

2. Preliminaries

Recall that for a group G and a topological space X, a G-action σ on X is a homomorphism from G to the group of continuous self-maps of X. We will be using σ to represent this map, that is, for $t \in G$, $t \mapsto \sigma^t$ where $\sigma^t : X \to X$ is continuous. We will be concerned with $\mathbb{Z}^d$ actions for fixed $d \in \mathbb{N}$, as the language of group actions is well suited to discussion of finite dimensional dynamical systems. If $f_1, \dots f_d$ are commuting maps on X, then $(t_1, \dots, t_d) \mapsto f_1^{t_1} \circ \cdots \circ f_d^{t_d}$ is a well-defined group action on X.

Much as the orbit of a point can be identified with the sequence $\{f^i(x)\}_{i\in\mathbb{N}}$, the $\mathbb{Z}^d$-orbit of a point $x \in X$ can be identified with the collection $\{\sigma^t(x)\}_{t\in\mathbb{Z}^d}$, as in Figure 1.

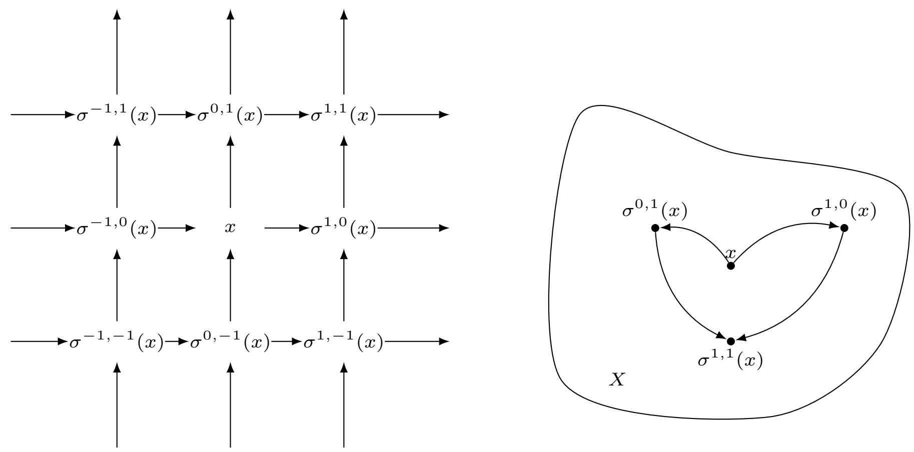


FIGURE 1. Orbit of a $\mathbb{Z}^2$ action as $\{\sigma^t\}_{t\in\mathbb{Z}^d}$ and as seen in X.

Multidimensional shift spaces serve as an interesting source of examples and as a focus of inquiry. Shift spaces have the benefit that the group action is quite clear. For a finite set Σ with the discrete topology, we will use $\Sigma^{\mathbb{Z}^d}$ to denote the space of maps from $\mathbb{Z}^d$ to Σ with the product topology. A point $x \in \Sigma^{\mathbb{Z}^d}$ is given by its coordinates, i.e. $x = \{x_i\}_{i\in\mathbb{Z}^d}$. For the purposes of this paper, for $t \in \mathbb{Z}^d$, we will define $|t| = \sum |t_i|$, i.e. the distance from t to the origin in the taxicab metric. Then, the topology on $\Sigma^{\mathbb{Z}^d}$ is consistent with the metric ρ given by $\rho(x, y) = 2^{-n}$ where $n = \min_{x_t \neq y_t}\{|t|\}$. The group action is as follows: for $t \in \mathbb{Z}^d$, $(\sigma^t(x))_s = x_{s+t}$.

Recall that for a $\mathbb{Z}$-action generated by iteration of a map f on a compact metric space (X, ρ), the **ω-limit set** of a point x is

$$\omega(x) = \bigcap_{i\in\mathbb{N}} \overline{\{f^{t+i}(x)|t \in \mathbb{N}\}}.$$

That is to say, it is the limit set of x under the action of the semigroup $\mathbb{N}$ in the sense of Gottschalk and Hedlund,[5]. Another formulation is

$$\omega(x) = \{y \in X|\forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z} \text{ such that } t > M \text{and } \rho(y, f^t(x)) < \epsilon\}.$$

This is the **forward limit set** of x under the group action. The **backwards limit set** is also of interest and is known as the **α-limit set** of x:

$$\alpha(x) = \bigcap_{i\in\mathbb{N}} \overline{\{f^{-(t+i)}(x)|t \in \mathbb{N}\}}.$$

This is the limit set of x under the action of the semigroup $-\mathbb{N}$, which can also be written

$$\alpha(x) = \{y \in X | \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z} \text{ such that } t < -M \text{and } \rho(y, f^t(x)) < \epsilon\}.$$

We are interested in studying limit behaviors in multi-dimensional dynamical systems. In particular, we would like to define analogues to the ω- and α-limit sets for a $\mathbb{Z}^d$-action σ on a compact metric space (X, ρ).

As a first consideration, there are many more directions to choose from in $\mathbb{Z}^d$. In $\mathbb{Z}$ there are essentially only two directions, in particular $1 \in \mathbb{Z}$ and $-1 \in \mathbb{Z}$. These are of course, generators for the semigroups $\mathbb{N}$ and $-\mathbb{N}$ respectively.

The natural generalization to $\mathbb{Z}^d$ actions is to consider d-dimensional vectors as our directions.

Definition 1. *The set of* ***directions in*** $\mathbb{Z}^d$ *is the set*

$$\mathcal{D}^{(d)} = \{\eta \in \mathbb{Z}^d | \gcd\{\eta_i\} = 1\}.$$

The restriction to those vectors whose entries are relatively prime is necessary to simplify several expressions later in the paper. It should be noted, however, that this restriction does not actually restrict the directions under consideration. We will use e_i for $i \in \{1, \ldots, n\}$ to denote the standard basis vectors for $\mathbb{Z}^d$. Note that for each $i \in \{1, \ldots, n\}$, $e_i \in \mathcal{D}^{(d)}$.

Now we define our notion of a directional limit set.

Definition 2. *For $\eta \in \mathcal{D}^{(d)}$ and $x \in X$, define the **η-type limit set** of x be*

$$L_\eta(x) = \{y \in X | \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \textit{ such that } t \cdot \eta > M \textit{and } \rho(y, \sigma^t(x)) < \epsilon\}$$

where $\cdot$ denotes the usual dot product of vectors.

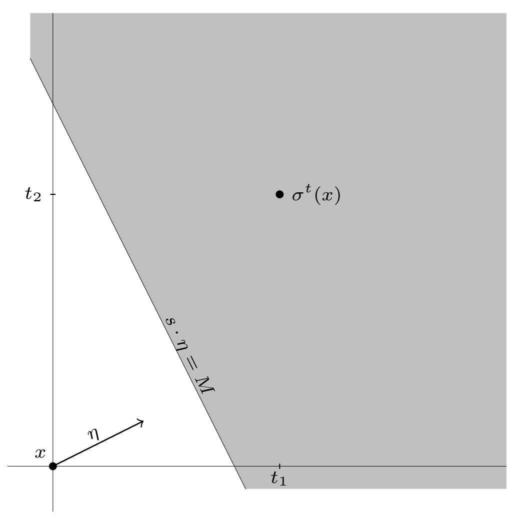


FIGURE 2. $L_\eta(x)$ in the orbit space for $\eta = (2, 1)$. The shaded region represents those images of x for which $t \cdot \eta > M$.

It should be noted that the set $G_\eta = \{t \in \mathbb{Z}^d | t \cdot \eta > 0\}$ is a semigroup and that $L_\eta(x)$ is the limit set of x under its action.

Remark 3. *In the case that* $d = 1$*, these are the familiar limit sets;* $L_{-1}(x) = \alpha(x)$ *and* $L_1(x) = \omega(x)$*.*

In [11], Piotr Oprocha studies a particular kind of limit set in these systems. In particular, each (strict) d-dimensional quadrant of $\mathbb{Z}^d$ is a semigroup, and he looks at the limit sets of points under these actions. We introduce the following definitions which generalize these limit sets.

Definition 4. *For* $E \subset \mathcal{D}^{(d)}$ *finite and* $x \in X$*, define the* E***-type limit sets of*** x *to be*

$$L_E^+(x) = \{y \in X | \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \text{such that } \min_{\eta \in E}\{t \cdot \eta\} > M \text{ and } \rho(y, \sigma^t(x)) < \epsilon\}.$$

and

$$L_E^-(x) = \{y \in X | \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \text{such that } \max_{\eta \in E}\{t \cdot \eta\} > M \text{ and } \rho(y, \sigma^t(x)) < \epsilon\}.$$

It should be noted that $L_E^+(x)$ is the limit set of x under the action of $\bigcap_{\eta \in E} G_\eta$. Note that this semigroup may be empty, in which case the limit set is empty as well.

Also note that the union of semigroups is in general not a semigroup, so the limit set of the form $L_E^-(x)$ are not (in general) semigroup limit sets in the sense of Gottschalk.

When $E = \{\eta\}$ is a singleton, these two limit sets coincide and are equal to L_η as defined above.

The limit sets of Oprocha are L_E^+ type limits where E is taken to be $\{a_i e_i\}_{1 \le i \le d}$ for a choice of $a_i \in \{-1, 1\}$ and where e_i denotes the ith unit basis vector in $\mathbb{Z}^d$.

Lemma 5. *For* $E \subset \mathcal{D}^{(d)}$ *finite, and* $x \in X$*, the sets* $L_{E^\pm}(x)$ *are closed and invariant under* σ^s *for all* $s \in \mathbb{Z}^d$*.*

Proof. We will demonstrate that $L_E^-(x)$ is closed and invariant. That $L_E^+(x)$ is closed and invariant can be proved in an identical fashion, but also follows from known results for limit sets under the action of semigroups [5].

Let $E \subseteq \mathcal{D}^{(d)}$ and $x \in X$. Let $\{p_i\}_{i \in \mathbb{N}}$ be a sequence of points in $L_E^-(x)$ which converges to the point $p \in X$. Now, let $M \in \mathbb{N}$ and $\epsilon > 0$. Pick $N \in \mathbb{N}$ such that for $n > N$, $\rho(p_n, p) < \epsilon/2$.

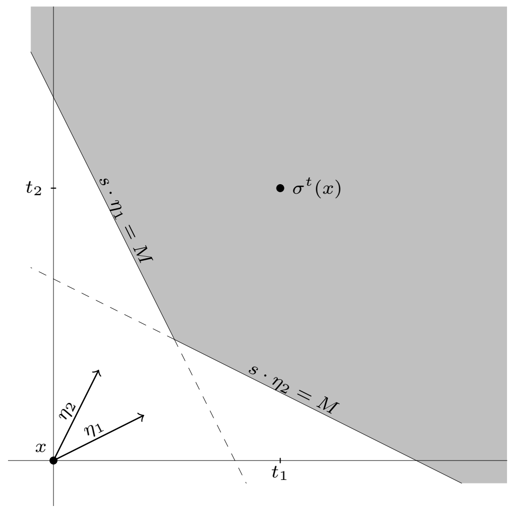
Since $p_n \in L_E^-(x)$, there exists $t \in \mathbb{Z}^d$ such that $\max_{\eta \in E}\{t \cdot \eta\} > M$ and $\rho(\sigma^t(x), p_n) < \epsilon/2$. In particular, $\max_{\eta \in E}\{t \cdot \eta\} > M$ and $\rho(\sigma^t(x), p) \le \rho(\sigma^t(x), p_n) + \rho(p_n, p) < \epsilon$. Thus $L_E^-(x)$ is closed.

Now let us see that it is invariant under σ^t for all $t \in \mathbb{Z}^d$. Let $t \in \mathbb{Z}^d$. Since E is finite, there exists $K \in \mathbb{N}$ such that $|t \cdot \eta| < K$ for all $\eta \in E$. Let $M \in \mathbb{N}$ and $\epsilon > 0$. Since σ^t is uniformly continuous, let $\delta > 0$ such that $\rho(a, b) < \delta$ implies $\rho(\sigma^t(a), \sigma^t(b)) < \epsilon$. Let $p \in L_E^-(x)$. and consider $p' = \sigma^t(p)$. Since $p \in L_E^-(x)$, there exists $s \in \mathbb{Z}^d$ such that $\max_{\eta \in E}\{s \cdot \eta\} > M + K$ and $\rho(\sigma^s(x), p) < \delta$. By our earlier note about t, $\max_{\eta \in E}\{s + t \cdot \eta\} > M$ and $\rho(\sigma^{s+t}(x), p') = \rho(\sigma^t(\sigma^s(x)), \sigma^t(p)) < \epsilon$. Thus $p' \in L_E^-(x)$. $\square$

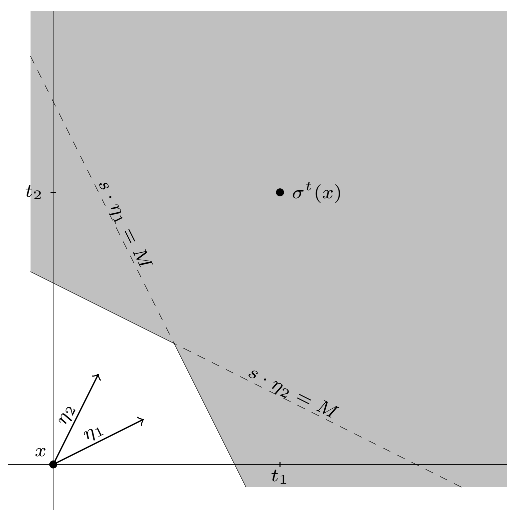
The following is the most general limit set one can define in this setting.

Definition 6. *For* $x \in X$*, define the* ω***-limit set of*** x *to be*

$$\omega(x) = \{y \in X | \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \text{such that } \max_{1 \le i \le d}\{|t_i|\} > M \text{ and } \rho(y, \sigma^t(x)) < \epsilon\}.$$



(a) $L_E^+(x)$



(b) $L_E^-(x)$

FIGURE 3. Limit sets for $E = \{\eta_1, \eta_2\}$. The shaded regions represents those images of x for which $\min_{\eta \in E}\{t \cdot \eta\} > M$ and $\max_{\eta \in E}\{t \cdot \eta\} > M$ respectively.

The proofs of the following results are straightforward.

Lemma 7. *Let $x \in X$. Then for $E = \{\pm e_i\}_{1 \leq i \leq d}$ where e_i denotes the ith unit basis vector in $\mathbb{Z}^d$,*

$$\omega(x) = \bigcup_{\eta \in \mathbb{Z}^d} L_\eta(x) = L_E^-(x).$$

Corollary 8. *The set $\omega(x)$ is closed and invariant under σ^s for all $s \in \mathbb{Z}^d$.*

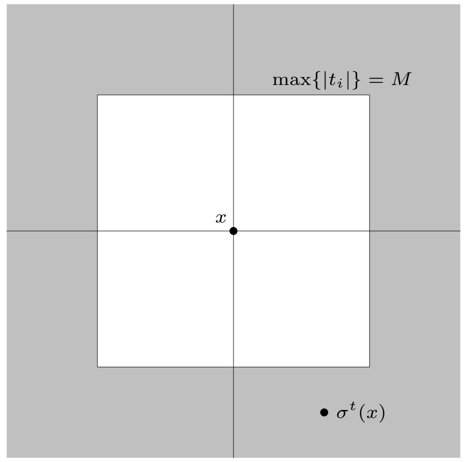


FIGURE 4. The limit set $\omega(x)$. The shaded regions represents those images of x for which $\max_{1\le i\le d}\{|t_i|\} > M$.

Theorem 9. *Let $E \subset \mathcal{D}^d$ finite and $x \in X$. For each $\zeta \in E$ we have the following:*

$$L_E^+(x) \subseteq \bigcap_{\eta\in E} L_\eta(x) \subseteq L_\zeta(x) \subseteq \bigcup_{\eta\in E} L_\eta(x) = L_E^-(x) \subseteq \omega(x)$$

Each of these inclusions is potentially strict. To see this, we consider the 2-dimensional full shift on two symbols, $\{0,1\}^{\mathbb{Z}^2}$ and describe points which demonstrate the ineuquality of the limit sets. Let $x \in \{0,1\}^{\mathbb{Z}^2}$ given by $x_t = 0$ if t is in quadrants one or three and $x_t = 1$ otherwise. For $E = \{(1,0),(0,1)\}$, $L_E^+ = \{(0)_{t\in\mathbb{Z}^2}\}$, whereas $\bigcap_{\eta\in E} L_\eta(x) = \{(0)_{t\in\mathbb{Z}^2}, (1)_{t\in\mathbb{Z}^2}\}$. The set $L_{\{(1,0)\}}$ contains points which are all ones below a horizontal line and all zeros above that horizontal line. The set $L_{\{(0,1)\}}$ has points which are all ones to the left of a vertical line and all zeros to the right. Finally, $\omega(x)$ has points which are all ones above a horizontal line and all zeros below it, as well as points which are zero to the left of a vertical line and one to the right.

Lemma 10. *For $E \subset \mathcal{D}^{(d)}$ finite, $\epsilon > 0$ and $x \in X$:*

(1) *there exists $m \in \mathbb{N}$ such that for all $t \in \mathbb{Z}^d$ with $\min_{\eta\in E}\{t\cdot\eta\} \ge m$, $\rho(\sigma^t(x), L_E^+(x)) < \epsilon$, and*
(2) *there exists $m \in \mathbb{N}$ such that for all $t \in \mathbb{Z}^d$ with $\max_{\eta\in E}\{t\cdot\eta\} \ge m$, $\rho(\sigma^t(x), L_E^-(x)) < \epsilon$.*

Proof. The proofs for each item are essentially identical, so we shall prove only the first.

Suppose to the contrary, i.e. for a given $\eta \in X$, $\epsilon > 0$ and $x \in X$, for each $m \in \mathbb{N}$ there exists $t_m \in \mathbb{Z}^d$ with $\min_{\eta\in E}\{t_m\cdot\eta\} \ge m$ and $\rho(\sigma^{t_m}(x), L_E^+(x)) > \epsilon$. By passing to a subsequence if necessary, suppose that the sequence $\sigma^{t_m}(x)$ converges to y. By the above, $\rho(y, L_E^+(x)) > \epsilon$. But this same sequence of t_m demonstrates that $y \in L_E^+(x)$. This contradicts the choice of t_m. $\square$

3. Generalizations of Internal Chain Transitivity

In one-dimensional systems given by a continuous function f on a compact metric space X, it has been shown that sets Λ which are ω-limits can be identified by internal properties, cf. [1], [2], [3].

In particular, a set $\Lambda \subseteq X$ is *internally chain transitive* if for each $\epsilon > 0$ and each pair $x, y \in \Lambda$, there exists an $n \in \mathbb{N}$ and a collection $\{x = x_0, x_1, \ldots, x_n = y\}$ of points in Λ such that for $0 \leq j < n$, $d(f(x_j), x_{j+1}) < \epsilon$. Such a collection of points is called an *ϵ-chain from x to y*. Finally, a set $\Lambda \subseteq X$ is *weakly incompressible* if $M \cap \overline{f(\Lambda \setminus M)} \neq \varnothing$ for each nonempty closed proper subset M of Λ.

It is fairly easy to show that these three concepts are related. Sarkovski showed that ω-limit sets are weakly incompressible. Hirsch later showed that ω-limit sets are also internally chain transitive, [6].

Furthermore, it is not difficult to show that in this context, if Λ is a closed, nonempty subset of X, then it is weakly incompressible if and only if it is internally chain transitive, [4].

While there are many possible ways to generalize these notions to the case of a $\mathbb{Z}^d$ action on a metric space X, we are most interested in a characterization of limit sets, and will be tailoring our exposition for this goal.

Remark 11. *For the purposes of the following definitions, we fix $E \subseteq \mathcal{D}^d$ finite and let F denote one of E^+ or E^-. Additionally, for purposes of brevity and clarity, for $t \in \mathbb{Z}^d$, we will let $||t||_F$ equal either $\min_{\eta \in E}\{t \cdot \eta\}$ or $\max_{\eta \in E}\{t \cdot \eta\}$ if F is E^+ or E^- respectively. Also, recall that for $t \in \mathbb{Z}^d$, we use $|t|$ to denote the distance of t from the origin using the taxicab metric, i.e. $|t| = \sum |t_i|$.*

Definition 12. *For $\epsilon > 0$, an **ϵ-F-mesh** is a collection $\{p_t\}_{M \leq ||t||_F \leq K}$ for some $M, K \in \mathbb{Z}$ with $M \leq K$ with the property that $\rho(\sigma^s(p_t), p_{t+s}) < \epsilon$ when $|s| = 1$, and $M \leq ||t||_F \leq ||t+s||_F \leq K$. An **$\epsilon$-$F$-band** is an ϵ-F-mesh for which $M = K$. An **ϵ-F-mesh (band) in a set** D is one for which each p_t belongs to D.*

This is a straightforward generalization of the one-dimensional notion of an ϵ-chain. It is at this point that the restriction of directions to the set $\mathcal{D}^d$ becomes relevant. In doing so, we ensure that ϵ-F-meshes are nontrivial for appropriate choices of M. In particular, when $F = E^-$ for a finite set of directions E, meshes are nonempty provided that $M \geq 0$. In contrast, for certain choices of E, E^+ meshes are empty for M sufficiently large, but this occurs exactly when L_E^+ limit sets are empty.

As a final observation, it is worth noting that E^+-meshes are infinite when they exist, whereas E^--meshes may be finite or infinite depending on the set E. Of particular note, if E is the set of the unit basis vectors and their additive inverses, then E^--meshes are finite for $M \geq 0$.

We now offer a first generalization of internal chain transitivity to the setting of $\mathbb{Z}^d$ actions.

Definition 13. *A set $\Lambda \subseteq X$ is **internally F-meshed** if for each $\epsilon > 0$ and each pair $x, y \in \Lambda$ there exists an ϵ-F-mesh $P = \{p_t\}_{M \leq ||t||_F \leq K}$ in Λ such that there exists t_0 with $||t_0||_F = M$ and $x = p_{t_0}$ and t_1 with $||t_1||_F = K$ and $p_{t_1} = y$.*

Indeed, for $d = 1$ and $F = \{1\}$ (and by uniform continuity $F = \{-1\}$ as well), this is precisely the property of internal chain transitivity.

Theorem 14. *If $A = L_F(z)$ for some $z \in X$, then A is internally F-meshed.*

Proof. Let $x, y \in L_F(x)$. Let $\epsilon > 0$ and by uniform continuity, choose $\delta > 0$ such that $\delta < \epsilon/2$ and if $\rho(p,q) < \delta$, $\rho(\sigma^t(p), \sigma^t(q)) < \epsilon/2$ for all $|t| \leq 1$. By Lemma 10, find $N \in \mathbb{N}$ such that for $||t||_F \geq N$, $\rho(\sigma^t(x), L_F(z)) < \delta$. Now, for all $t \in \mathbb{Z}^d$ with $||t||_F \geq N$, choose $p_t \in L_F(z)$ such that $\rho(\sigma^t(x), p_t) < \delta$. Since $x \in L_F(z)$, we can find some t_0 with $||t_0||_F \geq N$ such that we can choose $p_{t_0} = x$. Furthermore, since $y \in L_F(z)$, we can find t_1 with $||t_1||_F > ||t_0||_F$ such that we may choose $p_{t_1} = y$. By choice of δ, the collection $P = \{p_t\}_{||t_0||_F \leq ||t||_F \leq ||t_1||_F}$ is an ϵ-F-mesh with x and y in the required positions. Thus $L_F(z)$ is internally F-meshed. $\square$

An important distinction between chains and meshes is that chains are naturally transitive in the following sense. When $d = 1$, it is easy to check that chains can be concatenated to arrive at a longer chain. In particular, if $\{x_0, x_1, \ldots, x_n\}$ is an ϵ-chain from x to y and $\{y_0, y_1, \ldots, y_n\}$ an ϵ-chain from y to z, then $\{x_0, \ldots, x_n = y_0, \ldots y_n\}$ is an ϵ-chain from x to z.

However, meshes are not so easily concatenated. Firstly, it is not the case that a given ϵ-mesh may even by extended, much less extended in a particular manner. Examples of such meshes can easily be found in shifts of finite type. Furthermore, to concatenate meshes $P = \{p_t\}_{M \leq ||t||_F \leq K}$ and $Q = \{q_t\}_{K \leq ||t||_F \leq L}$ in an analogous way would require $p_t = q_t$ for all t with $||t||_F = K$.

The converse to Theorem 14 is known to be false, even in the case that $d = 1$, [2]. However, it has been demonstrated that there are many sets X for which the converse holds. In [4], some conditions are found which guarantee the converse. In their construction, the natural transitivity of chains is applied to generate a well behaved pseudo-orbit. We generalize these notions.

Definition 15. *For $\epsilon > 0$, an ϵ-F-**pseudo-orbit** is a collection $\{p_t\}_{M \leq ||t||_F}$ for some $M \in \mathbb{N}$ satisfying $\rho(\sigma^s(p_t), p_{t+s}) < \epsilon$ when $|s| = 1$ and $M \leq ||t||_F \leq ||t+s||_F$. An ϵ-F-pseudo-orbit in a set D is one for which $p_t \in D$ for all $||t||_F \geq M$.*

*An ϵ-F-pseudo-orbit $\{p_t\}_{M \leq ||t||_F}$ is **asymptotic** if for all $\delta > 0$ there exists a $K \in \mathbb{N}$ such that for $\{p_t\}_{L \leq ||t||_F}$ is a δ-F-pseudo-orbit.*

Remark 16. *Pseudo-orbits in shift spaces have particularly useful properties. Specifically, for $n \in \mathbb{N}$ and $\epsilon < 2^{-n}$, an ϵ-F-pseudo-orbit has the following property: if $s, t \in \mathbb{Z}^d$ such that $|s| \leq n$ and both p_t and p_{t+s} are defined, then $(p_t)_s = (p_{t+s})_0$, that is, the symbol of p_t in position s is the same as the symbol of p_{t+s} at the origin.*

Definition 17. *For $\epsilon > 0$, an ϵ-F-band C and $y \in X$, an ϵ-F-**mesh from** C **to** y is an ϵ-F-mesh $P = \{p_t\}_{K \leq ||t||_F \leq M}$ such that $\{p_t\}_{||t||_F = K} = C$ and there exists t with $||t||_F = M$ and $\rho(p_t, y) < \epsilon$.*

Definition 18. *A set $\Lambda \subseteq X$ is **internally mesh transitive with respect to** F (IMT-F) provided that there exist collections $\{\mathcal{C}_N\}_{N \in \mathbb{N}}$ of F-bands in Λ such that for all $\epsilon > 0$ there exists N_ϵ such that for $C \in C_{N_\epsilon}$ and $y \in X$ there exists an ϵ-F-mesh $P = \{p_t\}_{K \leq ||t||_F \leq M}$ in Λ from C to y and $C' = \{p_t\}_{||t||_F = M} \in C_{N_{\epsilon/2}}$.*

Note that this is a generalization of internal chain transitivity. In dimension one, when F is either $\{1\}$ or $\{-1\}$, F-bands are points, and ϵ-F-meshes are ϵ-chains. By taking $\mathcal{C}_i = \Lambda$ for each i, it is easy to see that Λ is internally mesh transitive with respect to F if and only if Λ is internally chain transitive.

Theorem 19. *If $A = L_F(x)$ for some $x \in X$, then A is IMT-F.*

Proof. Let $x \in X$. For each $t \in \mathbb{Z}^d$, choose $p_t \in \Lambda$ so as to minimize $\rho(\sigma^t(x), p_t)$.

For $i \in \mathbb{N}$, by uniform continuity, let $\delta_i < 2^{-i-1}$ such that if $\rho(p, q) < \delta_i$, then $\rho(\sigma^s(p), \sigma^s(q)) < 2^{-i-1}$ for all $|s| \leq 1$. Choose $N_i \in \mathbb{N}$ such that for all $s \in \mathbb{Z}^d$ with $||s||_F \geq N_i$, $\rho(\sigma^s(x), L_F(x)) < \delta_i$ as guaranteed by Lemma 10. Without loss of generality, we may assume that $N_i \leq N_{i+1}$ for all $i \in \mathbb{N}$. Notice that for each $t \in \mathbb{Z}^d$ such that $N_i \leq ||t||_F$, the p_t chosen earlier satisfies $\rho(\sigma^t(x), p_t) < \delta_i$. Let $\mathcal{C}_i = \{\{p_t\}_{||t||_F = K}\}_{K \geq N_i}$.

Now, for each $\epsilon > 0$ choose i such that $\epsilon > 2^{-i-1} > 0$. Let $C \in \mathcal{C}_i$ and $y \in L_F(x)$. Finally, choose j such that $\epsilon/2 > 2^{-j-1} > 0$. Then, by choice of δ_i, the collection $\{p_t\}_{N_i \leq ||t||_F \leq K}$ is an ϵ-F-mesh for all $K \geq N_i$. In particular, since $y \in L_F(x)$, there exists $t \in \mathbb{Z}^d$ with $||t||_F = M > N_j$ such that $\rho(y, \sigma^t(x)) < \epsilon$.

Then $P = \{p_t\}_{N_i \leq ||t||_F \leq M}$ is an ϵ-F-mesh from C to y and $\{p_t\}_{||t||_F = M} \in \mathcal{C}_j$.

So, we have now verified that the collection $\{\mathcal{C}_i\}$ witnesses the internal mesh transitivity of $L_F(x)$. □

As internal mesh transitivity is a generalization of internal chain transitivity, the converse of this theorem is not true for all spaces. However, if a set $\Lambda \subseteq X$ is internally mesh transitive, we can construct certain well behaved pseudo-orbits.

First, we note the following geometric property of Z^d.

Lemma 20. *For each F, there exists $D \in \mathbb{N}$ such that for $p, q \in \mathbb{Z}^d$ and $K \in \mathbb{N}$ such that $||p||_F < K < ||q||_F$ and $|p - q| = 1$, there exists $t \in \mathbb{Z}^d$ with $||t||_F = K$ and $|p - t| < D$ and $|q - t| < D$.*

Proof. Let us consider $\mathbb{Z}^d$ as a subset of $\mathbb{R}^d$. Let $K \in \mathbb{N}$. The set $\Delta_K = \{x \in \mathbb{R}^d | \; ||x||_F = K\}$ is a piecewise union of finitely many hyperplanes. In particular, it is easy to see that there exists a number $D_0 \in \mathbb{N}$ such that for all $x \in \Delta_K$, $B_{D_0}(x) \cap \{t \in \mathbb{Z}^d \mid ||t||_F = K\} \neq \varnothing$ where B_{D_0} is the ball of radius with the standard Euclidean metric on $\mathbb{R}^d$. Let $D \in \mathbb{N}$ such that $y \in B_{D_0+1}(x)$ (with the Euclidean metric) implies that $|x - y| < D$.

Now, let $p, q \in \mathbb{Z}^d$ satisfy the hypotheses of the lemma. Let t_0 be the unique point on Δ_K which is on the line segment from p to q. By the above, there is exists $t \in \mathbb{Z}^d$ with $||t||_F = K$ such that $t \in B_{D_0}(t_0)$. By the triangle inequality, $t \in B_{D_0+1}(p)$ and $t \in B_{D_0+1}(q)$. By choice of D, $|p - t| < D$ and $|q - t| < D$. □

As mentioned earlier, a pivotal property of ϵ-chains is that ϵ-chains which share an endpoint can be concatenated into a longer ϵ-chain. We will need a similar result for meshes, but the geometry of the required overlap is more complicated. The following Lemma allows for the necessary concatenation.

Lemma 21. *For $\epsilon > 0$ there exists $\xi > 0$ such that if $K > M$ and $P = \{p_t\}_{M \leq ||t||_F \leq K}$ is a ξ-F-mesh and $Q = \{q_t\}_{L \leq ||t||_F}$ is a $\xi/2$-F-pseudo-orbit with $L \leq K$ such that the $p_t = q_t$ when $||t||_F = K$, then $R = \{r_t\}_{M \leq ||t||_F}$ given by*

$$r_t = \begin{cases} p_t & ||t||_F \leq K \\ q_t & \textit{otherwise} \end{cases}$$

is an ϵ-F-pseudo-orbit.

Proof. Let $\epsilon > 0$.

Let $D \in \mathbb{N}$ as given by the previous Lemma. Let $\epsilon_0 = \epsilon$ and by uniform continuity, for $1 \leq i \leq a$, choose $\epsilon_i > 0$ such that $\epsilon_i < \epsilon_{i-1}/2$ and if $\rho(p, q) < \epsilon_i$

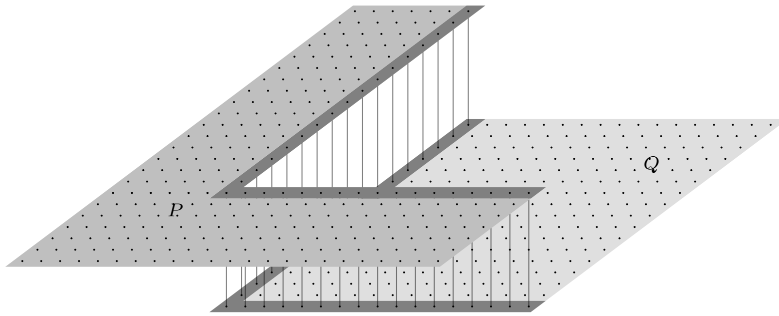


FIGURE 5. Schematic for construction of R with $F = \{(1,0),(0,1)\}^-$.

then $\rho(\sigma^s(p), \sigma^s(q) < \epsilon_{i-1}/2$ for all $s \in \mathbb{Z}^d$ with $|s| \leq 1$. We will now check that $\xi = \epsilon_{D+1}$ satisfies the hypotheses of the lemma.

Let $K > M$, $P = \{p_t\}_{M \leq ||t||_F \leq K}$ be a ξ-F-mesh and $Q = \{q_t\}_{K \leq ||t||_F}$ be a $\xi/2$-F-pseudo-orbit. Let $R = \{r_t\}$ be the collection defined above. We will now verify that R is an ϵ-F-pseudo-orbit.

Since ξ and $\xi/2$ are less than ϵ, we need only check that $\rho(\sigma^{s-t}(r_t), r_s) < \epsilon$ when $||t||_F < M$ but $||s||_F > M$ and $|s-t| = 1$. By the previous lemma, let $t_0 \in \mathbb{Z}^d$ such that $||t_0||_F = K$ and both $|s - t_0|$ and $|t - t_0|$ are less than D. By choice of ϵ_D and the triangle inequality, $\rho(\sigma^{s-t_0}(q_{t_0}), q_s) < \epsilon/2$ and that $\rho(\sigma^{t-t_0}(q_{t_0}), p_t) < \epsilon_1/2$. And thus, by choice of ϵ_1, $\rho(\sigma^{s-t}(r_t), r_s) < \epsilon$ as required. □

Definition 22. *The* ***limit set of an*** F***-pseudo-orbit*** $P = \{p_t\}_{K \leq ||t||_F}$ *is the set:*

$$L_F(P) = \{y \in X | \forall M \in \mathbb{N}, \epsilon > 0\ \exists t \in \mathbb{Z}^d \textit{such that} ||t||_F > M \textit{and } \rho(y, p_t) < \epsilon\}.$$

Theorem 23. *If* $\Lambda \subseteq X$ *is IMT-F and closed, then for all* $\epsilon > 0$ *there exists an asymptotic* ϵ*-*F*-pseudo-orbit* P *in* Λ *such that* $L_F(P) = \Lambda$.

Proof. Let $\Lambda \subseteq X$ be a closed set which is IMT-F. Since Λ is compact, for all $n \in \mathbb{N}$, let $\{x_i^n\}_{1 \leq i \leq k_n} \subseteq \Lambda$ such that $\Lambda \subseteq \bigcup_{1 \leq i \leq k_n} B_{2^{-n}}(x_i^n)$. For $j \in \mathbb{N}$, there is a unique $l \in \mathbb{N}$ and $1 \leq i \leq k_l$ such that $j = i + \sum_{n=0}^{l} k_n$. Define $x_j = x_i^{l+1}$.

For $N \in \mathbb{N}$ let $\mathcal{C}_N$ be the collections of F-bands witnessing that Λ is IMT-F. Let $\epsilon > 0$ and choose $\xi > 0$ for ϵ as in Lemma 21. Choose $C_0 \in \mathcal{C}_{N_\xi}$ and let $P_0 = \{p_t^0\}_{K_0 \leq ||t||_F \leq K_1}$ be a ξ-F-mesh from C_0 to x_1 with $C_1 = \{p_t^0\}_{||t||_F = K_1} \in \mathcal{C}_{N_{\xi/2}}$.

Let $n \in \mathbb{N}$ and assume $C_{n-1} \in \mathcal{C}_{N_{\xi/2^{n-1}}}$ and K_{n-1} have been defined. Let $P_{n-1} = \{p_t^{n-1}\}_{K_{n-1} \leq ||t||_F \leq K_n}$ be a $\xi/2^{n-1}$-F-mesh from C_{n-1} to x_n with $C_n = \{p_t^{n-1}\}_{||t||_F} = K_n \in \mathcal{C}_{N_{\xi/2^n}}$.

Now, define $P = \{p_t\}_{K_0 \leq ||t||_F}$ as follows. Observe that for all t such that $||t||_F \geq K_0$, there exists a unique $n \in \mathbb{N}$ such that $K_n \leq ||t||_F < K_{n+1}$. Define $p_t = p_t^n$.

Observe that by Lemma 21, this is in fact an ϵ-F-pseudo-orbit. We will now demonstrate that it is an asymptotic F-pseudo-orbit. For all $\epsilon' > 0$ let $0 < \xi' < \epsilon'$ as required by Lemma 21, and notice that there exists $n \in \mathbb{N}$ such that $\xi/2^n < \xi'$. Then, by construction $\{p_t\}_{t \in T_{K_n}}$ is a ϵ' pseudo-orbit. Thus P is an asymptotic pseudo-orbit as claimed.

Furthermore, since Λ is closed and P is an F-pseudo-orbit in Λ, $L_F(P) \subseteq \Lambda$. And finally, let $z \in \Lambda$. Let $\gamma > 0$ and $M \in \mathbb{N}$. Fix $j \in \mathbb{N}$ such that $\xi/2^{j+1} < \gamma$ and observe that for all $i > \sum_{n=0}^{j} k_n$ there exists $l > i$ such that $\rho(x_l, z) < \xi/2^{j+1} < \gamma$. In particular, take $i = \max\{1 + \sum_{n=0}^{j} k_n, M\}$. Notice that $K_l \geq M$ and that x_l is within $\xi/2^j$ of an element of the C_{K_l}. Thus $z \in L_F(P)$. □

In the study of $\mathbb{Z}^d$ actions, the notion of shadowing pseudo-orbits is particularly important.

Definition 24. *For $x \in X$ and $P = \{p_t\}_{M \leq ||t||_F}$, we say that x **ϵ-F-shadows** P if for all $||t||_F \geq M$, $\rho(\sigma^t(x), p_t) < \epsilon$.*

*We say that X has **F-shadowing** if for all $\epsilon > 0$ there exists $\delta > 0$ such that each δ-F-pseudo-orbit is ϵ-F-shadowed by some $x \in X$.*

Another notion that we make extensive use of is limit shadowing, as a generalization of the property of the same name from [4].

Definition 25. *For $x \in X$ and $P = \{p_t\}_{M \leq ||t||_F}$, we say that x **limit F-shadows** P if for all $\delta > 0$ there exists $K > M$ such that x δ-F-shadows $P' = \{p_t\}_{K \leq ||t||_F}$.*

*We say that X has **limit F-shadowing** if for all $\epsilon > 0$ there exists $\delta > 0$ such that each asymptotic δ-F-pseudo-orbit is limit shadowed by some $x \in X$.*

Theorem 26. *If X has limit F-shadowing, then a closed $\Lambda \subseteq X$ is IMT-F if and only if $\Lambda = L_F(x)$ for some $x \in X$.*

Proof. Let $x \in X$. By Lemma 5, $L_F(x)$ is closed. By Theorem 19, $L_F(x)$ is IMT-F.

Now, let Λ be a closed set which is IMT-F. By Theorem 23, for all $\epsilon > 0$ we can construct an asymptotic ϵ-F-pseudo-orbit $P = \{p_t\}_{M \leq ||t||_F}$ in Λ such that $L_F(P) = \Lambda$. Since X has limit shadowing, there exists $x \in X$ such that $\rho(\sigma^t(x), p_t) < \epsilon$ for all $||t||_F \geq M$ and for all $\xi > 0$ there exists $K > M$ such that $\rho(\sigma^t(x), p_t) < \xi$ for all $||t||_F \geq K$.

We now argue that $L_F(P) = L_F(x)$. Let $z \in L_F(x)$. Let $\epsilon > 0$ and $M \in \mathbb{N}$. Let $K > M$ such that $\rho(\sigma^t(x), p_t) < \epsilon/2$ for all $||t||_F \geq K$ by above. Choose $t \in \mathbb{Z}^d$ such that $||t||_F > K$ and $\rho(\sigma^t(x), z) < \epsilon/2$, Then $\rho(z, p_t) < \epsilon$, and $z \in L_F(P)$, so $L_F(x) \subseteq L_F(P)$. A similar argument verifies that $L_F(P) \subseteq L_F(x)$, and so the sets are equal. Thus, $\Lambda = L_F(x)$ as required. □

4. Results for Shift Spaces

We now turn our attention to the class of shift spaces. Recall that we use Σ to denote a finite alphabet and the **full shift** $\Sigma^{\mathbb{Z}^d}$ is the space of functions $p : \mathbb{Z}^d \to \Sigma$. A **shift space** is a closed σ-invariant subspace of the full shift. A **shift of finite type** is a shift space which is the complement of the union of all shifts of finitely many basic open sets, see [10, 12]. Shifts of finite type have been extensively studied in dimension one as well as in higher dimensions, [10].

Theorem 27. *Let Σ be a finite alphabet and $d \in \mathbb{N}$. Then for all F, $\Sigma^{\mathbb{Z}^d}$ has limit F-shadowing.*

Proof. Let $P = \{p_t\}_{M \leq ||t||_F}$ be an asymptotic F-pseudo-orbit in $\Sigma^{\mathbb{Z}^d}$. Fix $\alpha \in \Sigma$. Define $x \in \Sigma^{\mathbb{Z}^d}$ by $x_t = (p_t)_0$ (where 0 denotes the origin in $\mathbb{Z}^d$) if $||t||_F \geq M$ and $x_t = \alpha$ otherwise. It is simple to verify that x limit shadows P. □

Using an identical proof, we have the following.

Corollary 28. *Let Σ be a finite alphabet and $d \in \mathbb{N}$. Then for all F, $\Sigma^{\mathbb{Z}^d}$ has F-shadowing.*

Theorem 29. *A closed subset Λ of $\Sigma^{\mathbb{Z}^d}$ is IMT-F if and only if $\Lambda = L_F(x)$ for some $x \in \Sigma^{\mathbb{Z}^d}$.*

Proof. By Theorems 26, 19 and 27, this result is immediate. □

As previously mentioned, when $d = 1$, it has been shown that shifts of finite type also have limit shadowing, and hence the characterization of limit sets in the above sense. For $d > 1$, the situation is less clear.

The obstruction to generalizing this result for dimensions greater than one is the inherent difficulty of 'completing' an asymptotic pseudo-orbit in the way that Theorem 27 does for $\Sigma^{\mathbb{Z}^d}$. It is often difficult to tell whether such a completion exists in a given subshift and in fact, it is the case that certain asymptotic pseudo-orbits are not completable in this sense.

For example, let $\Sigma = \{0, 1\}$ and $d = 2$. Consider the shift of finite type X generated by forbidding patterns of the form

$$\begin{matrix} 1 & a & 1 \\ b & c & d \\ 1 & e & 1 \end{matrix}$$

where $a, b, c, d, e \in \{0, 1\}$. Now, let $\diagup$ be the element of X defined by

$$\diagup_t = \begin{cases} 1 & t = (n, n) \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

and $\diagdown$ the element of X defined by

$$\diagdown_t = \begin{cases} 1 & t = (n, -n) \text{ for some } n \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases}$$

Let $\mathbb{O}$ be the element of X which has zero as each of its coordinates.

Now, consider $F = \{t \in \mathcal{D}^d \mid |t| = 1\}$. Then the collection $P = \{p_t\}_{1 \leq ||t||_F}$ given by

$$p_{(x,y)} = \begin{cases} \sigma^{(0,y-x)}(\diagup) & xy > 0 \text{ and } y \geq x \\ \sigma^{(x-y,0)}(\diagup) & xy > 0 \text{ and } y \leq x \\ \sigma^{(0,y+x)}(\diagdown) & xy < 0 \text{ and } y \geq -x \\ \sigma^{(-y-x,0)}(\diagdown) & xy < 0 \text{ and } y \leq -x \\ \mathbb{O} & xy = 0 \end{cases}$$

is an asymptotic F-pseudo-orbit.

However, for $1 > \epsilon > 0$, a point $x \in \Sigma^{\mathbb{Z}^d}$ that ϵ-shadows P must agree with the central symbols of P as in Figure 6. In particular, x has as its initial square a pattern forbidden in X, specifically a 3 by 3 pattern whose corners are ones.

It should be noted that while P is not shadowed in X, we have not ruled out the possibility that X has limit F-shadowing. Indeed, the F-pseudo-orbit $P' = \{p_t\}_{2 \leq ||t||_F}$ is shadowed by a point in X as seen in Figure 7.

For F as chosen above, the question of completing the pseudo-orbit is a finite, and hence decidable question. For choices of F for which F-bands are infinite, the question is not finite and is related to tiling problems. The question on whether a given shift of finite type has limit F-shadowing for a particular F certainly bears further inquiry.

However, we can still say something in these cases.

$$\begin{array}{ccccccccccccccccccc}
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&\square&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&1&0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0&0&0&0&1&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0&0&0&0&0&1&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1&0\\
1&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{array}$$

FIGURE 6. Central symbols for the pseudo-orbit P.

Definition 30. *For $\epsilon > 0$ and a set X, an ϵ-F-mesh $P = \{p_t\}_{K \leq ||t||_F \leq M}$ is* ***compatible with*** *X provided that there is a collection $\{q_t\}_{||t||_F \leq M}$ in X with $q_t = p_t$ for $K \leq ||t||_F \leq M$ which satisfies $\rho(\sigma^s(q_t), q_{t+s}) < \epsilon$ when $|s| = 1$ and $||t||_F \leq ||t+s||_F \leq M$.*

Definition 31. *A set $\Lambda \subseteq X$ is* ***compatibly IMT-F*** *if it is IMT-F and there is a collection $\{\mathcal{C}_n\}$ that witnesses this has the property that for infinitely many $n \in \mathbb{N}$, there exists $C \in \mathcal{C}_n$ such that C is compatible with X.*

Notice that the compatibility condition in the above definition does not require that the elements of the extension lie in Λ, and thus this is not a property internal to Λ. However, we have the following result for a $\mathbb{Z}^d$-action on a compact metric space X.

Lemma 32. *For all $x \in X$, $L_F(x)$ is compatibly IMT-F.*

Proof. In the proof of Lemma 19, fix $K > N_1$. Then $C = \{x_t\}_{||t||_F = K}$ is compatible with X, as witnessed by the collection $Q = \{q_t\}_{||t||_F \leq K}$ where $q_t = p_t$ if $||t||_F \geq N_i$ and $q_t = \sigma^t(x)$ otherwise. ☐

Theorem 33. *Let X be a shift of finite type. A closed subset Λ of X is compatibly IMT-F if and only if $\Lambda = L_F(x)$ for some $x \in X$.*

Proof. By the previous Lemma, we need only demonstrate that if Λ is compatibly IMT-F, it is the F-limit set of some $x \in X$. Let $\{\mathcal{C}_i\}$ be the collection of F-bands that witness the compatible internal mesh transitivity of Λ.

Since X is a shift of finite type, fix $N \in \mathbb{N}$ large enough that each forbidden pattern is smaller than $N \times N$. Choose $\epsilon < 2^{-N}$. Choose $\xi > 0$ for ϵ as in Lemma 21.

Now, fix $C = \{c_t\}_{||t||_F = K} \in \mathcal{C}_{N_\xi}$, and construct an asymptotic F-pseudo-orbit $P = \{p_t\}_{K \leq ||t||_F}$ in Λ with $L_F(P) = \Lambda$ as in the proof of Theorem 23.

Let $Q = \{q_t\}_{||t||_F \leq K}$ be a collection that witnesses the compatibility of C with X. Now, define $R = \{r_t\}_{t \in \mathbb{Z}^d}$ given by $r_t = q_t$ if $||t||_F \leq K$ and $r_t = p_t$ if $||t||_F \geq K$.

$$
\begin{array}{ccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & & & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

(a) Central symbols for P'

$$
\begin{array}{ccccccccccccccccccc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array}
$$

(b) A point x which limit F-shadows P'

FIGURE 7

The collection R is an asymptotic ϵ-F-pseudo-orbit with the additional property that r_t is defined for all $t \in \mathbb{Z}^d$.

Now, let $x \in \Sigma^{\mathbb{Z}^d}$ be defined by $x_t = (r_t)_0$ where 0 denotes the origin in $\mathbb{Z}^d$. By our choice of N and Remark 16, we see that x belongs to X. Finally, as in the proof of Theorem 26, $L_F(x) = L_F(P) = \Lambda$. Thus, we have constructed $x \in X$ with $L_F(x) = \Lambda$. $\square$

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(J. Meddaugh) Department of Mathematics, Baylor University, Waco, TX 76798–7328,USA

E-mail address, J. Meddaugh: jonathan_meddaugh@baylor.edu

(B. E. Raines) Department of Mathematics, Baylor University, Waco, TX 76798–7328,USA

E-mail address, B. E. Raines: brian_raines@baylor.edu