SHADOWING AND ω-LIMIT SETS OF CIRCULAR JULIA SETS

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Abstract. In this paper we consider quadratic polynomials on the complex plane $f_c(z) = z^2 + c$ and their associated Julia sets, J_c. Specifically we consider the case that the kneading sequence is periodic and not an n-tupling. In this case J_c contains subsets that are homeomorphic to the unit circle, usually infinitely many disjoint such subsets. We prove that $f_c : J_c \to J_c$ has shadowing, and we classify all ω-limit sets for these maps by showing that a closed set $R \subseteq J_c$ is internally chain transitive if, and only if, there is some $z \in J_c$ with $\omega(z) = R$.

1. Introduction

Let $f_c(z) = z^2 + c$ with c chosen so that the kneading sequence of f_c is periodic, but not an n-tupling. The interesting dynamics of this map are carried on its Julia set, J_c. This is a compact, locally connected space which contains (usually infinitely many) circles.

In a series of papers Baldwin established a symbolic representation for these spaces and their dynamics and also for quadratic Julia sets which are dendrites, [1] and [2]. Baldwin begins by defining a non-Hausdorff *itinerary topology* on a space of symbols. Then he identifies certain periodic sequences, τ, which are candidates for kneading sequences of quadratic maps. It is important to note that in this context, the kneading sequence of the quadratic map is not the itinerary of the critical point, but rather is defined in terms of external rays. Precise definitions and discussion of kneading sequences for a quadratic map f_c can be found in [9] and [1]. Associated to each τ is a Hausdorff subspace $\mathcal{E}_\tau$, and, in the case that τ is the kneading sequence for some f_c, $\mathcal{E}_\tau$ is homeomorphic with J_c. Moreover in this case $f_c : J_c \to J_c$ is topologically conjugate to the shift map on $\mathcal{E}_\tau$. In the case that τ is not the kneading sequence of any f_c the associated subspace $\mathcal{E}_\tau$ is still defined and is referred to as an *abstract Julia set*. These spaces are all compact, locally connected, and usually contain many circles.

A set R is *internally chain transitive (ICT)* provided for every $\epsilon > 0$ and $x, y \in R$ there is an ϵ pseudo-orbit in R from x to y. The ICT property has been studied extensively. In many cases (such as shifts of finite type, dendritic Julia sets, topologically hyperbolic maps, and certain maps of the interval) it can be shown that a closed set R is ICT if, and only if, there is a point x with $\omega(x) = R$; however there are also many settings where this is not the case, (sofic shift spaces and certain maps of the interval), [3], [4], [5], [6], and [7]. It seems that the key difference between the two involves whether the dynamical system has *shadowing* which we will define precisely in Section Three.

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It is known that when the Julia set of a quadratic map is not connected, it is a Cantor set and the dynamics are conjugate to the shift map on $\{0,1\}^{\mathbb{N}}$. When the Julia set is connected, in all but some exceptional cases, it is locally connected, and thus is either a dendrite or contains topological circles. In the disconnected and dendritic Julia sets, the dynamics exhibit shadowing and ω-limit sets are characterized by the ICT property [3], [6].

In this paper, we consider the case where there is an attracting or parabolic periodic orbit (and hence the Julia set contains circles), and show that similar results hold in this context as well. Specifically we show that a closed subset R of J_c is internally chain transitive if, and only if, there is a point $z \in J_c$ with $\omega(z) = R$. We also prove that $f_c : J_c \to J_c$ has shadowing.

In the next section we give some preliminaries and background to the symbolic representation of circular Julia sets. We also prove an important result connecting the symbolic representation to the metric. In Section Three, we prove that these spaces have shadowing, and then we use this result in Section Four to prove the main result characterizing ω-limit sets for the circular Julia sets.

2. Preliminaries

In [2], Baldwin develops a topology on a non-Hausdorff space of itineraries inside of which one can find a natural subsets which are conjugate under the shift map to quadratic Julia sets which are dendrites. In [1], he develops a similar theory which similarly captures the dynamics of quadratic Julia sets whose kneading sequences are periodic but not n-tuplings. In this section, we give a brief summary of definitions and results from these two papers.

Definition 1. *Let Λ be the topological space with underlying set $\{0, 1, *\}^\omega$ and topology induced by the non-Hausdorff topology $\{\varnothing, \{0\}, \{1\}\{0, 1, *\}\}$ on each factor space. We refer to this topology on Λ as the **itinerary topology**.*

It is clear that Λ is not Hausdorff, although we will see that it has many Hausdorff subspaces. For a finite sequence $\alpha \in \{0, 1, *\}^{<\omega}$, we have the cylinder set of α in Λ given by

$$B^{\Lambda}_{\alpha} = \{\beta \in \Lambda : \forall i \leq len(\alpha)\ \beta_i \neq * \Rightarrow \beta_i = \alpha_i\}.$$

With this topology, Λ contains many shift invariant compact metric spaces, and in particular contains copies of each dendritic Julia set [2]. We will be using the notation that Baldwin develops in [1].

Definition 2. *A sequence $\tau = (\tau_i)_{i\in\omega} \in \Lambda$ is Λ-**acceptable** if and only if it satisfies*

1. *For all $n \in \omega$, $\tau_n = *$ if and only if $\sigma^{n+1}(\tau) = \tau$.*
2. *For all $n \in \omega$ with $\sigma^n(\tau) \neq \tau$, there exists $k \in \omega$ such that $\{\tau_{n+k}, \tau_k\} = \{0, 1\}$, i.e., there exist disjoint open sets $U, V \subset \Lambda$ with $\sigma^n(\tau) \in U$ and $\tau \in V$.*

*If τ is Λ-acceptable, then $\alpha \in \Lambda$ is (Λ, τ)-**consistent** if for all $n \in \omega$, $\alpha_n = *$ implies $\sigma^{n+1}(\alpha) = \tau$. A sequence α is called (Λ, τ)-**admissible** if it is (Λ, τ) consistent and for all $n \in \omega$ for which $\sigma^n(\alpha) \neq *\tau$, there exists $k \in \omega$ such that $\{\alpha_{n+k}, \tau_{k-1}\} = \{0, 1\}$.*

Sequences τ which are Λ-acceptable correspond to kneading sequences of dendrite maps. The sequences α which are (Λ, τ)-admissible are the itineraries of other points on the associated dendrite.

Definition 3. *For $\tau \in \Lambda$, let $\mathcal{D}_\tau = \{\alpha \in \Lambda : \alpha$ is (Λ, τ)-admissible$\}$.*

Baldwin then proves the following theorems about $\mathcal{D}_\tau$ in [2].

Theorem 4. *Let τ be Λ-acceptable. Then $\mathcal{D}_\tau$ is a shift-invariant self-similar dendrite.*

Theorem 5. *Let $f_x : \mathbb{C} \to \mathbb{C}$ by $f_c(z) = z^2 + c$. If the Julia set J_c of f_c is a dendrite, then there is a Λ-acceptable τ such that $f_c|_{J_c}$ is conjugate to $\sigma|_{\mathcal{D}_\tau}$.*

Barwell and Raines have studied the dynamics of these spaces in [6]. In particular, they prove the following results.

Theorem 6. *Let τ be Λ-acceptable. Then σ has shadowing on $\mathcal{D}_\tau$.*

Theorem 7. *Let τ be Λ-acceptable. Then $B \subseteq \mathcal{D}_\tau$ is closed and internally chain transitive if and only if $B = \omega(z)$ for some $z \in \mathcal{D}_\tau$.*

As an immediate corollary, these properties hold for quadratic Julia sets which are dendrites.

In order to capture the dynamics of Julia sets with periodic kneading sequences, a different itinerary space is required since removal of a single point will not separate the space enough to allow for the required uniqueness of itinerary[1]. To accomplish the required encoding, Baldwin uses a second wild card # as follows.

The factor space for this itinerary topology is given by $\{0, 1, *, \#\}$. For a sequence $\alpha \in \{0, 1, *, \#\}^{\leq\omega}$, let $\alpha\restriction_n = (\alpha_i)_{i=0}^n$ and for $i \in \{0, 1\}$, let $s_i(\alpha)$ be the sequence given by

$$(s_i(\alpha))_j = \begin{cases} \alpha_j & : \alpha_j \in \{0, 1\} \\ i & : \alpha_j = * \\ 1 - i & : \alpha_j = \# \end{cases}$$

and let $K(\alpha) = \{\alpha, s_0(\alpha), s_1(\alpha)\}$.

Definition 8. *Let Γ be the topological space with underlying set $\{0, 1, *, \#\}^\omega$ and topology given by the basis $\{B_\alpha : \alpha \in \{0, 1, *, \#\}^{<\omega}\}$ where $B_\alpha = \{\beta \in \Gamma : \beta\restriction_{len(\alpha)} \in K(\alpha)\}$.*

Notice that in this topology, for $\alpha \in \Gamma$, every neighborhood of α contains the set $K(\alpha)$. If α contains the symbol $*$ or $\#$, then this set contains three distinct elements, and thus Γ is not Hausdorff.

Definition 9. *A sequence $\tau \in \Gamma$ is Γ-**acceptable** provided it satisfies*

1. *$\tau = \overline{\alpha *}$ for some $\alpha \in \{0, 1\}^{<\omega}$.*
2. *For all $n \in \omega$, the sets $K(\sigma^n(\tau))$ and $K(\tau)$ are either disjoint or equal, i.e. $\sigma^n(\tau) = \tau$ or there exist disjoint open sets $U, V \subset \Gamma$ with $\sigma^n(\tau) \in U$ and $\tau \in V$.*

*If τ is Γ acceptable, then $\alpha \in \Gamma$ is (Γ, τ)-**consistent** provided that $\alpha_n \in \{*, \#\}$ if and only if $\sigma^{n+1}(\alpha) = \tau$. A (Γ, τ)-consistent $\alpha \in \Gamma$ is (Γ, τ)-**admissible** provided that for all $n \in \omega$, the set $K(\sigma^n(\alpha))$ is either disjoint from both of, or equal to one of, $K(*\tau)$ and $K(\#\tau)$, i.e. $\sigma^n(\alpha) \in \{*\tau, \#\tau\}$ or there exists disjoint open sets $U, V \subset \Gamma$ with $\sigma^n(\alpha) \in U$ and $\{*\tau, \#\tau\} \subseteq V$.*

As in the case of Λ, we wish to consider certain Hausdorff subspaces of Γ. In particular, we have the following analogue of $\mathcal{D}_\tau$.

Definition 10. *For $\tau \in \Gamma$, let $\mathcal{E}_\tau = \{\alpha \in \Gamma : \alpha$ is (Γ, τ)-admissible$\}$.*

In [1], Baldwin proves the following results about $\mathcal{E}_\tau$.

Theorem 11. *Let τ be Γ-acceptable. Then $\mathcal{E}_\tau$ is a locally connected shift invariant compact metric space on which σ is exactly two-to-one.*

An important tool that is used in establishing the properties of $\mathcal{E}_\tau$ and $\mathcal{D}_\tau$ is the existence of a continuous function from the consistent sequences onto the admissible sequences. Baldwin defines the following[1].

Definition 12. *Let τ be Λ-acceptable and let α be (Λ, τ)-consistent. Define $\chi_\tau^\Lambda(\alpha)$ to be the $\beta \in \mathcal{D}_\tau$ for which every neighborhood of β is a neighborhood of α.*

Let τ be Γ-acceptable and let α be (Γ, τ)-consistent. Define $\chi_\tau^\Gamma(\alpha)$ to be the $\beta \in \mathcal{E}_\tau$ for which every neighborhood of β is a neighborhood of α.

That such a β exists is not immediately clear, but the fact is demonstrated by Baldwin in [2] and [1].

As Baldwin notes, for most sequences α, $\chi_\tau^\Gamma(\alpha) = \alpha$. In fact it is only when there exists $n \in \omega$ with $\sigma^n(\alpha)$ indistinguishable from τ (in the topology of Γ) that $\chi_\tau^\Gamma(\alpha) \neq \alpha$.

The following result of Baldwin is Theorem 4.11 in [1].

Theorem 13. (Baldwin [1]) *Let $c \in \mathbb{C}$ and suppose that f_c has an attracting or parabolic periodic point. Let θ be one of the external angles of the parameter c and let τ be the kneading sequence of c. Suppose that τ is Γ-acceptable. Then $f_c|_{J_c}$ is conjugate to $\sigma|_{E_\tau}$.*

For a Γ-acceptable sequence τ, the spaces $\mathcal{E}_\tau$ and $\mathcal{D}_\tau$ are both defined and in fact the two are significantly related by the following function[1].

Definition 14. *For Γ-acceptable τ, define $\psi_\tau : \mathcal{E}_\tau \to \mathcal{D}_\tau$ such that for each $\alpha \in \{0,1\}^\omega$, $\psi_\tau(\chi_\tau^\Gamma(\alpha)) = \chi_\tau^\Lambda(\alpha)$.*

The function ψ_τ has the following properties [1], as well as many others.

Theorem 15. *Let τ be Γ-acceptable and let p be the period of τ. Then*

1. *If $\beta \in \mathcal{D}_\tau \setminus \mathcal{P}_\omega$, then $\psi_\tau^{-1}(\beta)$ is a singleton.*
2. *If $\beta \in \mathcal{P}_\omega$, then $C_\beta = \psi_\tau^{-1}(\beta)$ is homeomorphic to a circle.*

As $\mathcal{E}_\tau$ is a subset of a non-Hausdorff space, the metric on $\mathcal{E}_\tau$ is not immediately obvious. We will now demonstrate that the metric on $\mathcal{E}_\tau$ is consistent with a natural definition of distance observable on finite initial segments.

Definition 16. *Let τ be a Γ-acceptable sequence. Define $\mathcal{P}_n = \{\alpha \in \mathcal{E}_\tau : \alpha_i \in \{*, \#\}$ for some $i \leq n\}$, and $\mathcal{P}_\omega = \bigcup_{i \in \omega} \mathcal{P}_i$.*

Definition 17. *Let $x, y \in \mathcal{E}_\tau$. Then $x\restriction_n \sim y\restriction_n$ if and only if there exists $z \in \mathcal{P}_\omega$ with $\{x\restriction_n, y\restriction_n\} \subseteq \{z\restriction_n, s_0(z)\restriction_n, s_1(z)\restriction_n\}$.*

An important characteristic of $\mathcal{E}_\tau$ is that the set $\{*\tau, \#\tau\}$ separates $\mathcal{E}_\tau$. Then $\mathcal{E}_\tau = S_0 \cup S_1 \cup \{*\tau, \#\tau\}$ where $S_0 = \{\alpha \in \mathcal{E}_\tau : \alpha_0 = 0\}$ and $S_1 = \{\alpha \in \mathcal{E}_\tau : \alpha_0 = 1\}$.

Lemma 18. *Let $x, y \in \mathcal{E}_\tau$ with $x\restriction_n \sim y\restriction_n$ and $z \in \mathcal{P}_\omega$ witnessing this. Then there exist continua $C(x)$ and $C(y)$ in $\mathcal{E}_\tau$ containing $\{x, z\}$ and $\{y, z\}$ respectively such that for all $i \leq n$, there exist $j, k \in \{0,1\}$ with $\sigma^i C(x) \subseteq \overline{S_j}$ and $\sigma^i C(y) \subseteq \overline{S_k}$.*

Proof. First, let us observe that if $x \in \mathcal{P}_n$ then $x\restriction_n$ contains a $*$ or a $\#$ and so $x\restriction_n = z\restriction_n$ and it follows that $x = z$. Thus, $\{x\}$ is a continuum containing x and z satisfying the desired criterion.

Now, suppose that $x \notin \mathcal{P}_n$.

In [1], Baldwin demonstrates that for all $\beta \in \{0,1\}^{<\omega}$, the set B^τ_β which is the closure of $B_\beta \cap \mathcal{E}_\tau$ in $\mathcal{E}_\tau$ is connected. Furthermore, for such β, $\sigma(B^\tau_\beta) = B^\tau_{\sigma(\beta)}$.

Consider $\beta = x\restriction_n$. Then $\beta \in \{0,1\}^{<\omega}$ and B^τ_β is a continuum that contains x. Furthermore, it is clear that $\sigma^i(B^\tau_\beta) = B_\sigma(\beta)^\tau \subseteq \overline{S_{x_i}}$ for $i \leq n$. Thus, we need only check that $z \in B^\tau_\beta$.

If $x\restriction_n = z\restriction_n$, then $z \in B^\tau_\beta$ immediately. Otherwise, suppose without loss that $x\restriction_n = s_0(z)\restriction_n$.

We demonstrate that $z \in B^\tau_\beta$ be showing that each neighborhood in $\mathcal{E}_\tau$ of z intersects B^τ_β. To see this, fix $l \in \mathbb{N}$ and let $k \in \mathbb{N}$ such that $k \geq \max\{l, n\}$ and $\sigma^k(z) = *\tau$. Define z^l to be the point with $z^l_i = s(z_0)_i$ for $i \leq k$ and $\sigma^k(z^l) = \#\tau$. Then $z^l \in B_{z\restriction_l} \cap \mathcal{E}_\tau$ and $z^l \in B^\tau_\beta$. Thus $B_{z\restriction_l} \cap B^\tau_\beta$ for all $l \in \mathbb{N}$, and $z \in B^\tau_\beta$.

That such a continuum exists for the pair y, z follows similarly. $\square$

Notice that if $x \neq y$, at least one of $C(x)$ and $C(y)$ is nondegenerate.

To establish the relation between $x\restriction_n \sim y\restriction_n$ and the metric on $\mathcal{E}_\tau$, we use the following result from R.L. Moore [10].

Definition 19. *A continuum M has **property N** provided that for every $\epsilon > 0$ there exists a finite collection $\mathcal{G}$ of nondegenerate continua such that every subcontinuum of M of diameter greater than ϵ contains an element of $\mathcal{G}$.*

Moore proved the following.

Theorem 20. *A regular curve M does not have property N if and only if there exists an arc $A \subseteq M$ such that for each subarc $A' \subseteq A$ there exists an arc $B \subseteq M$ whose intersection with A consists of precisely its endpoints, one of which belongs to A'.*

Recall that a continuum X is a *regular curve* provided that any two points of X can be separated in X by a finite set [11].

Theorem 21. *Let τ be a Γ-acceptable sequence. Then $\mathcal{E}_\tau$ has property N.*

Proof. First, observe that $\mathcal{E}_\tau$ is a regular curve: each pair of points is separated by a finite set (in fact, by a set of at most two points).

Assume that $\mathcal{E}_\tau$ does not have property N. Then let A be the arc in $\mathcal{E}_\tau$ as guaranteed by Theorem 20 and $B \subset \mathcal{E}_\tau$ an arc with endpoints in A.

Since ψ_τ is continuous, $\psi_\tau(A)$ and $\psi_\tau(B)$ are either arcs or points in $\mathcal{D}_\tau$. Since $\mathcal{D}_\tau$ is a dendrite and hence uniquely arcwise connected, $\psi_\tau(B) \subseteq \psi_\tau(A)$. If $\psi_\tau(B)$ is nondegenerate, then $\psi_\tau(B) \cap \psi_\tau(A)$ contains a point x which is not an endpoint of $\psi_\tau(B)$ for which $\psi_\tau^{-1}(x)$ is a singleton as there are at most countably many points with non-degenerate pre-image. Then $\psi_\tau^{-1}(x) \in A \cap B$ and is not an endpoint of B. This is a contradiction, and hence $\psi_\tau(B)$ is degenerate.

In this case, $\psi_\tau(B) = \{\beta\}$ with $\beta \in \mathcal{P}_\omega$ and B is an arc on the circle C_β in $\mathcal{E}_\tau$ and A contains the complementary arc. Let $A' \subseteq A$ be a subarc contained in $C_\beta - B$. Let B' be an arc in $\mathcal{E}_\tau$ which intersects A only at its endpoints, one of which belongs to A'. Then using an argument similar to that above, $\psi_\tau(B') = \{\beta'\}$ for some $\beta' \in \mathcal{P}_\omega$. Since $\psi_\tau(B')$ intersects $\psi_\tau(A') = \{\beta\}$, we actually have $\psi_\tau(B') = \{\beta\}$

and so $B' \subseteq C_\beta$. Since $B' \cap A' \neq \varnothing$, B' meets A at more than its endpoints. Thus no arc meeting A only at its endpoints meets A', contradicting our assumption about A. Thus $\mathcal{E}_\tau$ has property N. □

Theorem 22. *Let τ be a Γ-acceptable sequence. Then the following hold:*

(1) *For $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}$ such that if $x, y \in \mathcal{E}_\tau$ and $x\restriction_{N_\epsilon} \sim y\restriction_{N_\epsilon}$ then $d(x,y) < \epsilon$.*
(2) *For $N \in \mathbb{N}$, there exists δ_N such that if $x, y \in \mathcal{E}_\tau$ with $d(x,y) < \delta_N$, then $x\restriction_N \sim y\restriction_N$.*

Proof. First, let us prove (1). Suppose to the contrary, that there exists $\epsilon > 0$ such that for each $n \in \mathbb{N}$ there exists $x^n, y^n \in \mathcal{E}_\tau$ with $x^n\restriction_n \sim y^n\restriction_n$ and $d(x^n, y^n) \geq \epsilon$.

Since $\mathcal{E}_\tau$ has Property N, by Theorem 21, let $\mathcal{G}$ be a finite collection of nondegenerate subcontinua of $\mathcal{E}_\tau$ such that every subcontinuum of $\mathcal{E}_\tau$ with diameter at least $\epsilon/2$ contains an element of $\mathcal{G}$.

For each pair x^n, y^n, let $C(x^n)$ and $C(y^n)$ be the subcontinua guaranteed in Lemma 18. Since $d(x^n, y^n) \geq \epsilon$, the diameter of one of these subcontinua must be at least $\epsilon/2$. Without loss of generality, assume that $C(x^n)$ has diameter at least $\epsilon/2$. Since there are finitely many elements of $\mathcal{G}$ and infinitely many $C(x^n)$, there exists an element $G \in \mathcal{G}$ such that $G \subseteq C(x^n)$ for infinitely many $n \in \mathbb{N}$.

In particular, for all $n \in \mathbb{N}$ there exists $i_n \in \{0,1\}$ with $\sigma^n(G) \subseteq \overline{S_{i_n}}$. Furthermore, since G is uncountable, $\{g \in G : \sigma^n(g) \in S_{i_n}\} = \{g \in G : g_n = i_n\}$ is uncountable. This violates the unique itinerary property of $\mathcal{E}_\tau$, giving us a contradiction.

For (2), let $N \in \mathbb{N}$. Observe that $\mathcal{P}_\omega$ is dense in $\mathcal{E}_\tau$. Consider the open cover $\mathcal{B} = \{B_{\beta\restriction_N} : \beta \in \mathcal{P}_\omega\}$. Since $\mathcal{E}_\tau$ is compact, let $\mathcal{B}'$ be a finite subcover and let $\delta_N > 0$ the Lebesgue number for this finite subcover. Then for $x, y \in \mathcal{E}_\tau$ with $d(x,y) < \delta_N$, there exists $z \in \mathcal{P}_\omega$ such that $x, y \in B_{z\restriction_N}$. This implies that $\{x\restriction_N, y\restriction_N\} \subseteq \{z\restriction_N, s_0(z)\restriction_N, s_1(z)\restriction_N\}$ and hence $x\restriction_N \sim y\restriction_N$. □

3. Shadowing

In this section we prove that σ has shadowing on $\mathcal{E}_\tau$. We will use this fact in the next section where we characterize ω-limit sets. First, let us recall what it means for a map to have shadowing.

Definition 23. *A map $f : X \to X$ has **shadowing** provided that for each $\epsilon > 0$ there exists $\delta > 0$ such that if (x_i) is a δ-pseudo-orbit for f then there exists a point $y \in X$ such that for all i, $d(f^i(y), x_i) < \epsilon$.*

Now, we begin with a technical lemma about the arrangement of points in P_ω.

Lemma 24. *Let τ be a Γ-acceptable sequence with period p. Let $x \in \mathcal{P}_{n-1}$ and $z \in \mathcal{P}_k$ with $K(\sigma^n(x)\restriction_S) \cap K(\sigma^n(z)\restriction_S) \neq \varnothing$. If $S + n > k + p$ and $S > p$, then $\sigma^n(x) = \sigma^n(z)$.*

Proof. Let x, z, n, S as in the statement of the Lemma. Let $m = \max\{n, k+1\}$. Then $S + n \geq m + p$ and $m \geq n$. Since

$$K(\sigma^n(x)\restriction_S) \cap K(\sigma^n(z)\restriction_S) \neq \varnothing,$$

we also have

$$K(\sigma^m(x)\restriction_p) \cap K(\sigma^m(z)\restriction_p) \neq \varnothing.$$

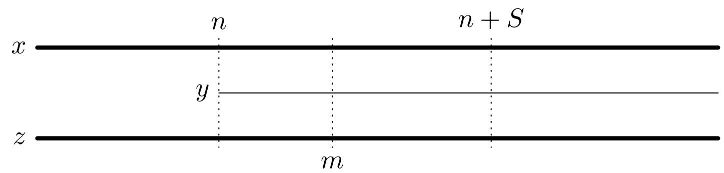

FIGURE 1. Schematic of x and z, with $y \in K(\sigma^n(x)\restriction_S) \cap K(\sigma^n(z)\restriction_S)$.

Since $x, z \in \mathcal{P}_{m-1}$, there exist $i, j \in \mathbb{N}$ with $\sigma^m(x) = \sigma^i(\tau)$ and $\sigma^m(z) = \sigma^j(\tau)$. Since both $\sigma^m(x)$ and $\sigma^m(z)$ are periodic with period p, it follows that for all $i \in \mathbb{N}$

$$K(\sigma^m(x)\restriction_i) \cap K(\sigma^m(z)\restriction_i) \neq \varnothing.$$

Since τ is Γ-acceptable, in fact we have $\sigma^m(x) = \sigma^m(z)$. If $m = n$, we are done. Otherwise, $m = k+1$. But in this case, we have

$$K(\sigma^n(x)\restriction_{(k+1-n)}) \cap K(\sigma^n(z)\restriction_{k+1-n}) \neq \varnothing$$

and $\sigma^{k+1}(x) = \sigma^{k+1}(z)$. Combining these agreements, we see that $\sigma^n(x) = \sigma^n(z)$. □

Now we are ready to prove shadowing on $\mathcal{E}_\tau$.

Theorem 25. *Let τ be a Γ-acceptable sequence. Then the map σ on $\mathcal{E}_\tau$ has shadowing.*

Proof. Let $\tau = \overline{\tau_1\tau_2\cdots\tau_{p-1}*}$ be a Γ-acceptable sequence. Let $\epsilon > 0$ and choose N_ϵ according to Lemma 22. Fix $M = N_\epsilon + 4p + 2$ and choose $R = M + 1$. Let $\delta_R > 0$ as in Lemma 22.

Let (x^i) be a δ_R-pseudo-orbit in $\mathcal{E}_\tau$. Thus, for all $i \in \mathbb{N}$ there exists $z \in \mathcal{P}_\omega$ with $\{\sigma(x^i)\restriction_R, x^{i+1}\restriction_R\} \subseteq K(z\restriction_R)$.

To construct a point w which ϵ-shadows (x^i), we wish to first define (possibly finite) sequences (z^j) in $\mathcal{E}_\tau$, $(m_j) \in \mathbb{N} \cup \{\infty\}$ and $(f_j) \in \mathbb{N} \cup \{\infty\}$ with the following properties:

(1) For each j, m_j is the greatest element of $\mathbb{N} \cup \{\infty\}$ for which there exists $z \in \mathcal{E}_\tau$ satisfying $x^i\restriction_M \in K(\sigma^i(z)\restriction_M)$ for $m_{j-1} < i \leq m_j$.
(2) z^j satsifies the above property.
(3) For each j with $f_j < \infty$, $z^j \in \mathcal{P}_{f_j} \setminus \mathcal{P}_{f_j - 1}$.
(4) $\sup\{m_j\} = \infty$.
(5) For all j, $f_j \leq m_j$
(6) If $m_j < \infty$, then $m_j + N_\epsilon + 1 < f_{j+1}$

Let us assume that we have such a sequence and construct our shadowing point. For each $j \in \mathbb{N}$ with m_j finite, $m_j \leq m_j + N_\epsilon + 1 < f_{j+1}$ and so there is a unique $n_j \in \{0,1\}$ such that $\sigma^{m_j+1}(s_{n_j}(z^j)\restriction_{N_\epsilon}) = z^{j+1}\restriction_{N_\epsilon}$.

Now, define w as follows. For $i \in \mathbb{N}$ choose $j \in \mathbb{N}$ with $m_{j-1} < i \leq m_j$. If $m_j < \infty$, define $\hat{w}_i = s_{n_j}(z^j)_i$. If $m_j = \infty$, let $\hat{w}_i = z^j_i$ and $\hat{w} = (\hat{w}_i)$. Then, let $w = \chi^\Gamma_\tau(\hat{w})$. That w shadows (z^i) is not difficult to see. Pick $k \in \mathbb{N}$. By construction, if $m_{j-1} < k \leq m_j$, then $x^k\restriction_{N_\epsilon}$ and $\sigma^k(w)\restriction_{N_\epsilon}$ are both elements of $K(\sigma^k(z^j)\restriction_{N_\epsilon}$, and thus $x^k\restriction_{N_\epsilon} \sim \sigma^k(w)\restriction_{N_\epsilon}$, and so $d(x^k, \sigma^k(w)) < \epsilon$.

Now, let us construct the required sequences.

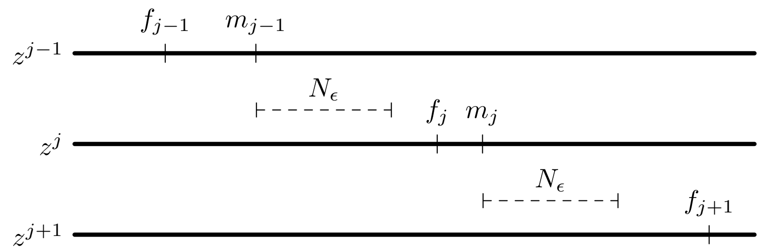

FIGURE 2. Schematic of z^{j-1}, z^j and z^{j+1}

First, we define z^1 and m_1 as follows. Consider the set Z_1 containing all $m \in \mathbb{N} \cup \{\infty\}$ for which there exists $z \in \mathcal{E}_\tau$ with $x^i\restriction_M \in K(\sigma^i(z)\restriction_M)$ for $i \leq m$. Notice that this collection is not empty, as $x^1 \in \mathcal{E}_\tau$ satisfies the property for $m = 1$. Let m_1 be the supremum of these m. We will see that it is actually attained.

Suppose that there does not exists a column f for which $\{x_i^{f-i} : 0 \leq i \leq M\}$ contains more than one element. Then $z^1 = \chi_\tau^\Gamma((x_0^i)_{i\in\omega})$ satisfies the required property for $m_1 = \infty$ and $f_1 = \infty$.

Otherwise, let f be the first column for which $\{x_i^{f-i} : 0 \leq i \leq M\}$ is not a singleton. Then any $z \in \mathcal{E}_\tau$ satisfying the property belongs to $\mathcal{P}_f$ or else has $m \leq f$. However, $z = x_0^1 x_0^2 \cdots x_0^{f-1} * \tau$ is an element of $\mathcal{P}_f$ with $m \geq f$, so we need only consider elements of $\mathcal{P}_f$ in our argument. As there are only finitely many elements of $\mathcal{P}_f$, define $z^1 \in \mathcal{P}_f$ that attains m_1.

Let $f_1 \in \mathbb{N}$ minimal such that $z^1 \in \mathcal{P}_{f_1}$. We want $f_1 \leq m_1$. To see this, suppose that $f_1 > m_1$, and let $z^* \in \mathcal{P}_\omega$ with $\{\sigma(x^{m_1})\restriction_R, x^{m_1+1}\restriction_R\} \subseteq K(z^*\restriction_R)$. Then we can easily see that $z = x_0^1 x_0^2 \cdots x_0^{m_1} z^*$ satisfies $x^i\restriction_M \in K(\sigma^i(z)\restriction_M)$ for $i \leq m_1 + 1$, contradicting maximality of m_1. Thus $f_1 \leq m_1$ as desired.

Now, let $j \in \mathbb{N}$ and suppose that $f_j \leq m_j < \infty$ and z^j have been defined. We will now define m_{j+1}, z^{j+1} and f_{j+1}.

Consider the set Z_{j+1} containing all $m \in \mathbb{N} \cup \{\infty\}$ for which $m > m_j$ and there exists $z \in \mathcal{E}_\tau$ with $x^i\restriction_M \in K(\sigma^i(z)\restriction_M)$ for $m_j < i \leq m$. As before, let $m_{j+1} \in \mathbb{N} \cup \{\infty\}$ be the supremum over such m.

Observe that if there are no columns f greater than m_j for which $\{x_i^{f-i} : 0 \leq i \leq M\}$ contains more than one element, then the point z^j defined by $\chi_\tau^\Gamma((x_0^i)_{i=0}^\infty)$ satisfies the required property with $m_{j+1} = \infty$ and $f_{j+1} = \infty$.

Otherwise, an argument identical to that for the $j = 1$ case, we can find $f_{j+1} \leq m_{j+1}$ with $z_{j+1} \in \mathcal{P}_{f_j} \setminus \mathcal{P}_{f_{j+1}-1}$ which attains m_{j+1}.

Thus we can construct our sequences (m_j), (z^j) and (f_j). Since $m_j < m_{j+1}$ unless $m_j = \infty$, we see $\sup\{m_j\} = \infty$. All that remains is to verify that $m_j + N_\epsilon + 1 < f_{j+1}$

Let $j \in \mathbb{N}$ and suppose that $f_{j+1} \leq m_j + N_\epsilon + 1$.

Since (x^i) is a δ_R-pseudo-orbit, there exists $z^* \in \mathcal{P}_\omega$ with

$$\{\sigma(x^{m_j})\restriction_R, x^{m_j+1}\restriction_R\} \subseteq K(\sigma^{m_j+1}(z)\restriction_N).$$

In particular, since $M < R$ we then have

$$\sigma(x^{m_j})\restriction_{M-1} \in K(\sigma^{m_j+1}(z^*)\restriction_{M-1}) \cap K(\sigma^{m_j+1}(z^j)\restriction_{M-1}).$$

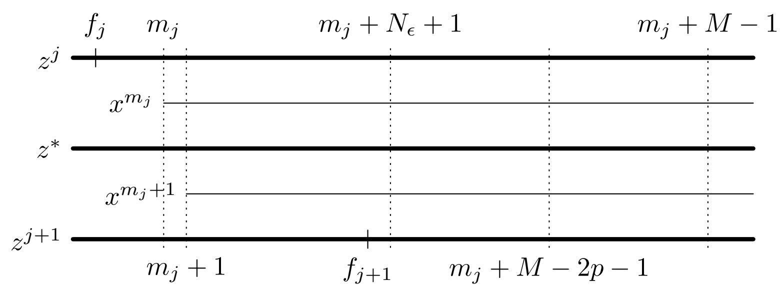

FIGURE 3. Schematic for z^j, z^{j+1} under the assumption $f_{j+1} \leq m_j + N_\epsilon + 1$.

Suppose that $z^* \in \mathcal{P}_{m_j+M-2p-1}$. Then z^j and z^* satisfy the hypotheses of Lemma 24, and so $\sigma^{m_j+1}(z^j) = \sigma^{m_j+1}(z^*)$

As we also have $x^{m_j+1}\restriction_{M-1} \in K(\sigma^{m_j+1}(z^*)\restriction_{M-1}) \cap K(\sigma^{m_j+1}(z^{j+1})\restriction_{M-1})$, applying Lemma 24 to z^* and z^{j+1} gives us $\sigma^{m_j+1}(z^{j+1}) = \sigma^{m_j+1}(z^*) = \sigma^{m_j+1}(z^j)$. As such, we see that z^j will actually satisfy $x^i\restriction_M \in K(\sigma^i(z^j)\restriction_M)$ for $m_{j-1} < i \leq m_{j+1}$, contradicting our choice of m_j.

Otherwise, we have that $z^* \notin \mathcal{P}_{m_j+M-2p-1}$. In this case, we have

$$K(\sigma^{m_j+1}(z^*)\restriction_{N_\epsilon+2p+1}) = \{\sigma^{m_j+1}(z^*)\restriction_{N_\epsilon+2p+1}\}.$$

This then gives us that

$$\sigma(x^{m_j})\restriction_{N_\epsilon+2p+1} \in K(\sigma^{m_j+1}(z^j)\restriction_{N_\epsilon+2p+1}) \cap K(\sigma^{m_j+1}(z^{j+1})\restriction_{N_\epsilon+2p+1}).$$

Here, we apply Lemma 24 to z^j and z^{j+1} and conclude that $\sigma^{m_j+1}(z^{j+1}) = \sigma^{m_j+1}(z^j)$. As such, we again see that z^j will actually satisfy $x^i\restriction_M \in K(\sigma^i(z^j)\restriction_M)$ for $m_{j-1} < i \leq m_{j+1}$, contradicting our choice of m_j.

In either case, we have arrived at a contradiction, so it must be the case that $m_j + N_\epsilon + 1 < f_{j+1}$ as requires. □

As an immediate corollary due to Theorem 13, we have the following:

Corollary 26. *Let $c \in \mathbb{C}$ and suppose that $f_c(z) = z^2 + c$ has an attracting or parabolic periodic point and kneading sequence τ which is not an n-tupling. Then the map f_c has shadowing on J_c.*

4. Classification of ω-limit sets

In this section we use the shadowing result of the previous section to prove that a set R is an ω-limit set of a point in J_c if, and only if it is ICT.

Definition 27. *A set $\Lambda \subseteq X$ is **internally chain transitive (ICT)** for $f : X \to X$ provided that for all $\epsilon > 0$ and for all $x, y \in \Lambda$, there exists a finite ϵ-pseudo-orbit in Λ from x to y.*

It is well known that ω-limit sets are internally chain transitive [8]. But the converse is not true for every space, [3].

Theorem 28. *Let τ be a Γ-acceptable sequence. Then a nonempty closed subset of $\mathcal{E}_\tau$ is an ω-limit set if and only if it is ICT.*

Proof. As mentioned above, we need only demonstrate that every nonempty closed internally chain transitive set Λ is an ω-limit set.

To that end, let $\Lambda \subseteq \mathcal{E}_\tau$ be nonempty, closed and internally chain transitive. Since Λ is closed and hence compact, for each $n \in \mathbb{N}$ let $\{y_1^n, \dots y_{k_n}^n\} \subseteq \Lambda$ be a set which 2^{-n} covers Λ. As Λ is internally chain transitive, for each $n \in \mathbb{N}$, we can find a 2^{-n} pseudo-orbit in Λ through $\{y_1^n, \dots y_{k_n}^n, y_1^{n+1}\}$. By concatenating these pseudo-orbits, we have a sequence (x^i) in Λ with the following properties.

(1) For all $n \in \mathbb{N}$, $\{x^i : i \geq n\}$ is dense in Λ.
(2) For all $\epsilon > 0$, there exists $n \in \mathbb{N}$ such that $(x^i)_{i\geq n}$ is an ϵ-pseudo-orbit.

In ordert to construct a point w with $\Lambda = \omega(w)$, we will mimic the construction in the previous theorem with one exception. Rather than fixing ϵ and N_ϵ at the outset, we will allow them to increase as we proceed.

First, for each $k \in \mathbb{N}$, let $N_k \in \mathbb{N}$, $M_k = N_k + 4p + 2$ and $R_k = M_k$ maximal such that $(x^i)_{i\geq k}$ is a δ_{R_k}-pseudo-orbit. Observe that (N_k) is non-decreasing and tends to ∞.

Now, we will want (possibly finite) sequences (z^j) in $\mathcal{E}_\tau$, $(m_j) \in \mathbb{N} \cup \{\infty\}$, $(f_j) \in \mathbb{N} \cup \{\infty\}$ with the following properties:

(1) For each j, m_j is the greatest element of $\mathbb{N} \cup \{\infty\}$ for which there exists $z \in \mathcal{E}_\tau$ satisfying $x^i\restriction_{M_i} \in K(\sigma^i(z)\restriction_{M_i})$ for $m_{j-1} < i \leq m_j$.
(2) z^j satsifies the above property.
(3) For each j with $f_j < \infty$, $z^j \in \mathcal{P}_{f_j} \setminus \mathcal{P}_{f_j-1}$.
(4) $\sup\{m_j\} = \infty$.
(5) For all j, $f_j \leq m_j$
(6) If $m_j < \infty$, then $m_j + N_{m_j} + 1 < f_{j+1}$

The existence of such sequences is guaranteed by the previous proof by replacing N_ϵ, M and R with the appropriate N_i, M_i and R_i. In cases in which two different indices are used, simply use the lesser of the two options.

Now, define w as follows. For $i \in \mathbb{N}$ choose $j \in \mathbb{N}$ with $m_{j-1} < i \leq m_j$. If $m_j < \infty$, define $\hat{w}_i = s_{n_j}(z^j)_i$. If $m_j = \infty$, let $\hat{w}_i = z_i^j$ and $\hat{w} = (\hat{w}_i)$. Then, let $w = \chi_\tau^\Gamma(\hat{w})$.

Let $\epsilon > 0$ and $N_\epsilon \in \mathbb{N}$ by Lemma 22. Choose $K \in \mathbb{N}$ such that if $k \geq K$, $N_k \geq N_\epsilon$. Then for $k \geq K$, there exists $j \in \mathbb{N}$ with $m_{j-1} < k \leq m_j$, and we see that $x^k\restriction_{N_k}$ and $\sigma^k(w)\restriction_{N_k}$ are both elements of $K(\sigma^k(z^j)\restriction_{N_k}$. Since $N_k \geq N_\epsilon$, we have $d(x^k, \sigma^k(w)) < \epsilon$.

Thus, for all $\epsilon > 0$, there exists K such that $\sigma^K(w)$ ϵ-shadows $(x^i)_{i\geq K}$. Immediately, we see that for all $\epsilon > 0$, $\omega(w) \subseteq B_\epsilon(\Lambda)$ and since Λ is closed, we have $\omega(w) \subseteq \Lambda$.

Finally, let $y \in \Lambda$ and let $\epsilon > 0$. Fix K such that $\sigma^K(w)$ $\epsilon/2$-shadows $(x^i)_{i\geq K}$. By construction of (x^i), there exists a subsequence (n_i) such that $n_1 \geq K$ and $(x^{n_i}) \subseteq B_{\epsilon/2}(y)$. Then $\sigma^{n_i}(w) \in B_\epsilon(y)$ for all $i \in \mathbb{N}$, and so $\omega(w) \cap B_\epsilon(y) \neq \varnothing$. Since this holds for all $\epsilon > 0$ and $\omega(w)$ is closed, $y \in \omega(w)$.

Thus $\omega(w) = \Lambda$ as desired. $\square$

As an immediate corollary, once again applying Theorem 13, we have the following result about Julia sets.

Corollary 29. *Let $c \in \mathbb{C}$ and suppose that $f_c(z) = z^2 + c$ has an attracting or parabolic periodic point and kneading sequence τ which is not an n-tupling. Then a nonempty closed subset of J_c is an omega limit set if and only if it is ICT.*

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