

INDECOMPOSABILITY IN INVERSE LIMITS WITH SET-VALUED FUNCTIONS

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ABSTRACT. In this paper, we develop a sufficient condition for the inverse limit of upper semi-continuous functions to be an indecomposable continuum. This condition generalizes and extends those of Ingram and Varagona. Additionally, we demonstrate a method of constructing upper semi-continuous functions whose inverse limit has the full projection property.

1. INTRODUCTION

A topological space X is a *continuum* if it is a nonempty compact, connected, metric space. Unless otherwise specified, all spaces in this paper are assumed to be continua. For a continuum X , we denote the collection of nonempty compact subsets of X by 2^X . A continuum Y which is a subset of X is a subcontinuum of X . For brevity, we will use $Y \leq X$ to indicate that Y is a subcontinuum of X .

If X and Y are continua, a function $f : X \rightarrow 2^Y$ is *upper semi-continuous at x* provided that for all open sets V in Y which contain $f(x)$, there exists an open set U in X with $x \in U$ such that if $t \in U$, then $f(t) \subseteq V$. If $f : X \rightarrow 2^Y$ is upper semi-continuous at each $x \in X$, we say that f is *upper semi-continuous (usc)*. A usc function $f : X \rightarrow 2^Y$ is called *surjective* provided that for each $y \in Y$, there exists $x \in X$ with $y \in f(x)$.

The *graph* of a function $f : X \rightarrow 2^Y$ is the subset $G(f)$ of $X \times Y$ for which $(x, y) \in G(f)$ if and only if $y \in f(x)$. Ingram and Mahavier showed that a function $f : X \rightarrow 2^Y$ is usc if and only if $G(f)$ is closed in $X \times Y$ [4]. This condition is easier to verify than the definition, and will be used frequently throughout the paper. As a consequence, it is easy to check that if $f : X \rightarrow 2^Y$ is usc and surjective, then the *inverse of f* , $(f^{-1} : Y \rightarrow 2^X)$ defined by $x \in f^{-1}(y)$ if and only if $y \in f(x)$ is well-defined and usc.

As a generalization of the well-studied and well-understood theory of inverse limits on continua, Ingram and Mahavier introduced the notion of inverse limits with set-valued functions [4, 5].

Definition 1. Let X_1, X_2, \dots be a sequence of continua and for each $i \in \mathbb{N}$, let $f_i : X_{i+1} \rightarrow 2^{X_i}$ be an upper semi-continuous function. The **inverse limit of the pair** $\{X_i, f_i\}$ is the set

$$\varprojlim \{X_i, f_i\} = \{(x_i)_{i=1}^{\infty} : x_i \in f_i(x_{i+1}) \text{ for all } i \in \mathbb{N}\}$$

with the topology inherited as a subset of the product space $\prod_{i=1}^{\infty} X_i$.

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The spaces X_i are called the *factor spaces* of the inverse limit, and the usc functions f_i the *bonding functions*. An *inverse sequence* $\{X_i, f_i\}$ denotes sequences X_i and f_i for which $f_i : X_{i+1} \rightarrow 2^{X_i}$ is a usc function. We will use $\pi_j : \varprojlim \{X_i, f_i\} \rightarrow X_j$ to denote the restriction to $\varprojlim \{X_i, f_i\}$ of the usual projection map on $\prod_{i=1}^{\infty} X_i$. For a subset L of the natural numbers, we will use π_L to denote projection from $\varprojlim \{X_i, f_i\}$ to $\prod_{i \in L} X_i$. We will often use the notation $[k, n]$ to denote the subset of the natural numbers consisting of k, n and all natural numbers between them.

As this is a generalization of the usual notion of inverse limits with continuous functions as bonding maps, it is natural to investigate the extent to which results carry over into this new context. Ingram and Mahavier pioneered this inquiry [4, 5], and many others have continued the investigation. We state some of these results below.

Theorem 2. [4] *Let $\{X_i, f_i\}$ be an inverse sequence. Then $\varprojlim \{X_i, f_i\}$ is non-empty and compact.*

It is well-known that if the bonding maps in an inverse sequence are continuous functions, the resulting inverse limit will be connected. This is, however, not the case in the context of usc functions. In particular, examples of non-connected inverse limits may be found in [4, 5] among others. The following known results indicate some sufficient conditions for an inverse limit of usc functions to be connected.

Theorem 3. [4] *Let $\{X_i, f_i\}$ be an inverse sequence and suppose that for each $i \in \mathbb{N}$ and each $x \in X_{i+1}$, $f_i(x)$ is connected. Then $\varprojlim \{X_i, f_i\}$ is connected.*

Theorem 4. [4] *Let $\{X_i, f_i\}$ be an inverse sequence and suppose that for each $i \in \mathbb{N}$ and each $x \in X_i$, $f_i^{-1}(x)$ is non-empty and connected. Then $\varprojlim \{X_i, f_i\}$ is connected.*

The following theorem is as stated in [3]. A more general version can be found in [8].

Theorem 5. [8] *Suppose \mathcal{F} is a collection of usc functions from $[0, 1]$ to $2^{[0, 1]}$ such that for every $g \in \mathcal{F}$ and $x \in [0, 1]$, $g(x)$ is connected, and that f is the function whose graph is the union of all the graphs of the functions in \mathcal{F} . If f is surjective and $G(f)$ is a continuum, then $\varprojlim f$ is a continuum.*

For functions defined as in this theorem, we will write, $f = \bigcup_{g \in \mathcal{F}} g$. The following theorem will also be useful.

Theorem 6. [8] *Suppose X is a compact Hausdorff continuum, and $f : X \rightarrow 2^X$ is a surjective upper semi-continuous set valued function. Then $\varprojlim f$ is connected if and only if $\varprojlim f^{-1}$ is connected.*

In their development of connectedness theorems, Ingram and Mahavier use a generalized notion of the graph of a function. We will use the following notation for this idea in this paper.

Definition 7. *Let $\{X_i, f_i\}$ be an inverse sequence. Then for $k < n$, define*

$$G_{[k, n]} \{X_i, f_i\} = \{(x_i)_{i=1}^{\infty} \in \prod_{i=1}^{\infty} X_i : x_i \in f_i(x_{i+1}) \text{ } k \leq i < n\}$$

and

$$G'_{[k,n]} \{X_i, f_i\} = \{(x_i)_{i=k}^n \in \prod_{i=k}^n X_i : x_i \in f_i(x_{i+1}) \text{ } k \leq i < n\}.$$

When no ambiguity will arise, the inverse sequence will be suppressed in the notation, and we will simply write $G_{[k,n]}$ or $G'_{[k,n]}$.

It is not difficult to see that $G'_{[k,n]}$ is a compact subset of $\prod_{i=k}^n X_i$, and that $G'_{[n,n+1]}$ is the graph of f_n^{-1} . Furthermore,

$$G_{[k,n]} = \prod_{i=1}^{k-1} X_i \times G'_{[k,n]} \times \prod_{i=n+1}^{\infty} X_i.$$

Additionally,

$$\varprojlim \{X_i, f_i\} = \bigcap_{n=1}^{\infty} \left(G'_{[1,n]} \times \prod_{i=n+1}^{\infty} X_i \right) = \bigcap_{n=1}^{\infty} (G_{[1,n]})$$

and

$$G'_{[k,n]} \{X_i, f_i\} = \pi_{[k,n]} \varprojlim \{X_i, f_i\}.$$

The next result appears in [4], and follows immediately from this observation.

Theorem 8. [4] *Let $\{X_i, f_i\}$ be an inverse sequence. Then $G'_{[1,n]}$ is connected for all $n \in \mathbb{N}$ if and only if $\varprojlim \{X_i, f_i\}$ is connected.*

Notice that this immediately implies that for $\varprojlim \{X_i, f_i\}$ to be connected, the graph of each bonding function must also be connected. There are many other results concerning connectedness of a usc inverse limit. For example, Ingram and Nall have developed some sufficient conditions in [1] and [7], respectively.

A property of considerable interest in continuum theory is that of indecomposability. A continuum X is *decomposable* provided that there exist proper subcontinua A, B of X with $A \cup B = X$. A continuum which is not decomposable is *indecomposable*.

There are several known results providing sufficient conditions for an inverse limit with single-valued bonding maps to be indecomposable. The main result of this paper is a generalization of the following well-known theorem (Theorem 2.7 of [6]).

Definition 9. *A single-valued map $f : X \rightarrow Y$ is **indecomposable** provided that for any pair $A, B \trianglelefteq X$ with $A \cup B = X$, then either $f(A) = Y$ or $f(B) = Y$.*

Theorem 10. *Let $\{X_i, f_i\}$ be an inverse sequence with indecomposable single-valued bonding maps. Then $\varprojlim \{X_i, f_i\}$ is indecomposable.*

A key element of the proof of this theorem is the fact that in an inverse limit with single-valued maps, a subcontinuum $K \trianglelefteq \varprojlim \{X_i, f_i\}$ for which $\pi_i(K) = X_i$ for infinitely many $i \in \mathbb{N}$ is, in fact, equal to $\varprojlim \{X_i, f_i\}$. This is yet another on the long list of properties that do not hold for inverse limits with usc functions.

Definition 11. *An inverse sequence $\{X_i, f_i\}$ has the **full projection property (fpp)** provided that if $K \trianglelefteq \varprojlim \{X_i, f_i\}$ satisfies $\pi_i(K) = X_i$ for infinitely many $i \in \mathbb{N}$, then $K = \varprojlim \{X_i, f_i\}$.*

For examples of usc functions for which the inverse sequence fails to have fpp, see [2] and [3].

Ingram and Varagona [2, 9] have both found conditions in which a usc inverse sequence has indecomposable inverse limit. We state the definitions and results below for completeness.

Ingram's theorem is a generalization of the notion of a two-pass map. A usc function $f : X \rightarrow 2^Y$ satisfies the two-pass condition provided that there are mutually exclusive connected open sets U and V of X for which $f|_U$ and $f|_V$ are single-valued maps and $\overline{f(U)} = \overline{f(V)} = Y$. Recall that a continuum X is a *simple n -od* if there is a point $p \in X$ such that X is the union of n arcs, each pair of which has p as their only common point.

Theorem 12. [2] *Let $\{X_i, f_i\}$ be an inverse sequence such that for all $i \in \mathbb{N}$, X_i is a simple n -od for some n and f_i satisfies the two-pass condition. If $\{X_i, f_i\}$ has fpp and $\varprojlim \{X_i, f_i\}$ is connected, then $\varprojlim \{X_i, f_i\}$ is indecomposable.*

In [9], Varagona defines a class of usc functions for which the inverse limit is indecomposable. Let $a \in (0, 1)$ and let $g, h : [0, 1] \rightarrow [0, 1]$ be single-valued functions satisfying the following: g is non-decreasing, $g(0) = 0$, $g(1) = a$, $g((0, 1)) = (0, a)$ and h is non-increasing, $h(0) = 1$, $h(1) = a$, $h((0, 1)) = (a, 1)$. A usc function $f : [0, 1] \rightarrow 2^{[0, 1]}$ is a *steeple with turning point a* provided that $G(f^{-1}) = G(g) \cup G(h)$.

Theorem 13. [9] *Let $\{X_i, f_i\}$ be an inverse sequence such that each f_i is a steeple. Then $\varprojlim \{X_i, f_i\}$ is indecomposable.*

In fact, Varagona proves that each such inverse limit is homeomorphic to the well-known bucket-handle continuum .

2. INDECOMPOSABLE USC FUNCTIONS

Our goal in this section is to generalize Theorem 10 in a way which also generalizes Theorems 12 and 13. As standing assumptions, all spaces X and Y are continua, and all usc functions f are surjective with connected graph.

The following will be our generalization of the concept of an indecomposable map.

Definition 14. *A usc function $f : X \rightarrow 2^Y$ is **indecomposable** provided that for any pair $A, B \trianglelefteq G(f)$ with $A \cup B = G(f)$, then either $\pi_Y(A) = Y$ or $\pi_Y(B) = Y$.*

This is one of several possible ways to generalize the notion of indecomposable from functions to usc functions. It is important to note that an indecomposable function does not necessarily result in an indecomposable inverse limit without additional conditions (Theorem 19). The following lemma verifies that this definition is consistent with the original.

Lemma 15. *Let $f : X \rightarrow Y$ be an indecomposable single-valued function. Then $f' : X \rightarrow 2^Y$ defined by $f'(x) = \{f(x)\}$ is indecomposable as a usc function.*

Proof. Let $f : X \rightarrow Y$ be an indecomposable function and let f' be defined as stated. Notice that by construction, $G(f|_C) = G(f'|_C)$ for all subsets C of X .

Let $A, B \trianglelefteq G(f')$ with $A \cup B = G(f')$. Then $\pi_X(A)$ and $\pi_X(B)$ are subcontinua of X whose union is X . Since f is indecomposable, one of $f(\pi_X(A))$ or $f(\pi_X(B))$ is equal to Y .

But f' is single valued, so $A = G(f|_{\pi_X(A)})$ and $B = G(f|_{\pi_X(B)})$. Thus $\pi_Y(A) = f(\pi_X(A))$ and $\pi_Y(B) = f(\pi_X(B))$. In particular, one of $\pi_Y(A)$ and $\pi_Y(B)$ is equal to Y , and so f' is an indecomposable usc function. \square

Indecomposability of a usc function is not difficult to check.

Example 16. The usc function $f : [0, 1] \rightarrow 2^{[0,1]}$ with graph shown in Figure 1 is indecomposable.

To see this, suppose that $A, B \trianglelefteq G(f)$ with $A \cup B = G(f)$. If one of A or B contains both $(0, 0)$ and $(1, 1)$, then that continuum is equal to $G(f)$, and has Y -projection equal to Y . Otherwise, we can assume without loss that $(0, 0) \in A$ and $(1, 1) \in B$. Since $A \cup B = G(f)$, it follows that $(a, 1)$ belongs to one of A or B . If it belongs to A then A contains an arc from $(0, 0)$ to $(a, 1)$, and so $\pi_Y(A) = Y$. If $(a, 1) \in B$, then $(b, 0) \in B$ as well, and so B contains an arc from $(b, 0)$ to $(1, 1)$, and in this case, $\pi_Y(B) = Y$. Thus, one of $\pi_Y(A)$ and $\pi_Y(B)$ is equal to Y , and so f is indecomposable.

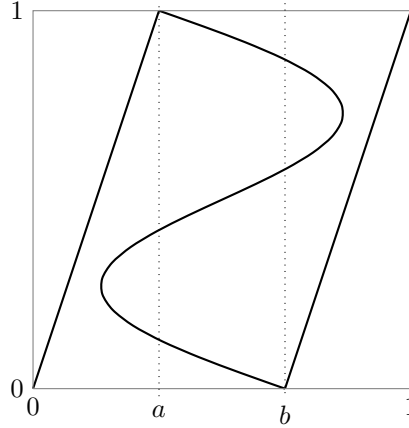


FIGURE 1. An example of the graph of an indecomposable usc function f on $[0, 1]$.

More examples of indecomposable usc functions appear in Section 4.

Lemma 17. Let $f : X \rightarrow 2^Y$ be a usc function satisfying the two-pass condition. If X is a simple n -od for some $n \in \mathbb{N}$, then f is indecomposable.

Proof. Let X be a simple n -od and let $f : X \rightarrow 2^Y$ be a usc function satisfying the two-pass condition. Let U and V be the mutually exclusive connected open subsets of X on which $f|_U$ and $f|_V$ are single-valued mappings with $\overline{f(U)} = \overline{f(V)} = Y$ that witness the two-pass condition.

Let $A, B \trianglelefteq G(f)$ with $A \cup B = G(f)$. Since $f|_U$ is single-valued, it is clear that if U is contained in $\pi_X(A)$, then $G(f|_U) \subseteq A$, and hence $\pi_Y(A) \supseteq \pi_Y(G(f|_U))$, and so $\pi_Y(A) \supseteq \overline{f(U)} = Y$. Similarly, if $U \subseteq \pi_X(B)$, or V is contained in $\pi_X(A)$ or $\pi_X(B)$, then one of $\pi_Y(A)$ or $\pi_Y(B)$ is equal to Y .

Thus, to establish that f is indecomposable, we need only verify that one of U or V is contained in one of $\pi_X(A)$ or $\pi_X(B)$. Since $A \cup B = G(f)$, and $A, B \trianglelefteq G(f)$, we know that $\pi_X(A)$ and $\pi_X(B)$ are subcontinua of X whose union is equal to X .

Since U and V are connected disjoint subsets of the n -od X , there is a point $t \in X$ for which U and V lie in different components of $X \setminus \{t\}$. Since X is a simple n -od, the closure of at least one of these components is an arc in X . Let I be this arc, and suppose without loss of generality, that $U \subseteq I$.

Let $u \in I$ be the endpoint of I distinct from t . Since $A \cup B = X$, let us assume that $u \in A$. If $t \in A$, then $U \subseteq I \subseteq A$. If $t \notin A$, then $A \subseteq I$, and so $V \subseteq X \setminus \{I\} \subseteq B$. If u were in B , a similar argument would hold.

Thus, $f : X \rightarrow 2^Y$ is indecomposable. \square

Lemma 18. *Let $f : [0, 1] \rightarrow 2^{[0, 1]}$ be a steeple with turning point a . Then f is indecomposable.*

Proof. Let f be a steeple function with turning point a . Let A, B be subcontinua of $G(f)$ whose union is equal to $G(f)$. If one of A or B contains both $(0, 0)$ and $(1, 0)$, then that subcontinuum is in fact equal to $G(f)$ and we are done. So, without loss of generality, assume $(0, 0) \in A$ and $(1, 0) \in B$.

Since $A \cup B = G(f)$, the point $(a, 1)$ belongs to at least one of A or B . If $(a, 1) \in A$, then A contains an arc from $(0, 0)$ to $(a, 1)$, and hence $\pi_Y(A) = Y$. Similarly, if $(a, 1) \in B$, then $\pi_Y(B) = Y$. Thus f is indecomposable. \square

Theorem 19. *Let $\{X_i, f_i\}$ be an inverse sequence for which each f_i is indecomposable. If $\{X_i, f_i\}$ has fpp and $\varprojlim \{X_i, f_i\}$ is connected, then $\varprojlim \{X_i, f_i\}$ is indecomposable.*

Proof. Let $\{X_i, f_i\}$ be an inverse sequence for which each bonding function is indecomposable. Furthermore, suppose that $\{X_i, f_i\}$ has fpp and $\varprojlim \{X_i, f_i\}$ is connected.

Let $A, B \trianglelefteq \varprojlim \{X_i, f_i\}$ with $A \cup B = \varprojlim \{X_i, f_i\}$. Then, for each $i > 1$, the projections $\pi_{\{i, i+1\}}(A)$ and $\pi_{\{i, i+1\}}(B)$ are subcontinua of $G'_{[i, i+1]}$ for which $\pi_{\{i, i+1\}}(A) \cup \pi_{\{i, i+1\}}(B) = G'_{[i, i+1]}$. As observed earlier, $G'_{[i, i+1]}$ is the graph of f_i^{-1} . Since f_i is indecomposable, it follows that one of $\pi_i(A)$ or $\pi_i(B)$ is equal to X_i .

Since this holds for all $i > 1$, it follows that for some $Z \in \{A, B\}$, $\pi_i(Z) = X_i$ for infinitely many $i \in \mathbb{N}$. Since $\{X_i, f_i\}$ has the full projection property, $Z = \varprojlim \{X_i, f_i\}$. Thus one of A or B is equal to $\varprojlim \{X_i, f_i\}$, and so $\varprojlim \{X_i, f_i\}$ is indecomposable. \square

Notice that by Lemma 17, Theorem 12 is a corollary of this theorem. We will see shortly that Theorem 13 is also a corollary of this theorem.

Theorem 10 is sometimes stated as a characterization of indecomposable continua, in the sense that every indecomposable continuum can be written as an inverse limit of an inverse sequence with indecomposable single-valued maps. This follows immediately from the facts that the identity map on an indecomposable continuum is indecomposable, and that $\varprojlim \{X, id_X\}$ is homeomorphic to X . In light of Lemma 15, a similar statement holds for usc functions.

Corollary 20. *A continuum X is indecomposable if and only if there exists an inverse sequence $\{X_i, f_i\}$ with indecomposable usc bonding functions and fpp for which $X \cong \varprojlim \{X_i, f_i\}$.*

That Theorem 13 is a corollary of Theorem 19 is not so immediate. To see this, we will establish that inverse sequences $\{X_i, f_i\}$ for which each f_i is a steeple all

have fpp and connected inverse limits. That these inverse limits are continua is straightforward.

Lemma 21. *Let $\{X_i, f_i\}$ be an inverse sequence for which each f_i is a steeple. Then $\varprojlim \{X_i, f_i\}$ is connected.*

Proof. If f_i is a steeple, then there is a nonincreasing map $g : [0, 1] \rightarrow [0, 1]$ and a nondecreasing map $h : [0, 1] \rightarrow [0, 1]$ with $G(f_i^{-1}) = G(g) \cup G(h)$. It follows that for all $x \in [0, 1]$, $f(x)$ is connected. So, by Theorem 3, $\varprojlim \{X_i, f_i\}$ is connected. \square

To see that an inverse sequence of steeple functions has fpp requires significantly more work. The next section is devoted to developing conditions which guarantee that an inverse sequence of usc functions has fpp. Additionally, these results will allow us to construct examples of an inverse sequence $\{X_i, f_i\}$ with $\varprojlim \{X_i, f_i\}$ an indecomposable continuum to which neither Theorem 12 nor Theorem 13 apply.

3. THE FULL PROJECTION PROPERTY

Recall that given an inverse sequence $\{X_i, f_i\}$ where each $f_i : X_{i+1} \rightarrow 2^{X_i}$ is an upper semi-continuous set-valued function, the notation $G_{[1,n]}$ refers to the subset of $\prod_{i=1}^{\infty} X_i$ where $x_i \in f_i(x_{i+1})$ for all $1 \leq i < n$. When we wish to discuss a similar set but as a subset of $\prod_{i=1}^n X_i$, we use the notation $G'_{[1,n]}$. Specifically,

$$G'_{[1,n]} = \{(x_i)_{i=1}^n \in \prod_{i=1}^n X_i : x_i \in f_i(x_{i+1}), 1 \leq i < n\}.$$

Recall that a continuum X is *irreducible between a and b* provided that no proper subcontinuum of X contains both a and b . Given two subsets A and B of X , we say that X is *irreducible between A and B* provided that no proper subcontinuum of X intersects both A and B .

Theorem 22. *Let $\{X_i, f_i\}$ be an inverse sequence where for each $i \in \mathbb{N}$, $f_i : X_{i+1} \rightarrow 2^{X_i}$ is usc and $\varprojlim \{X_i, f_i\}$ is a continuum. If for each $n \in \mathbb{N}$, there exist points $a, b \in X_n$, so that $G'_{[1,n]}$ is irreducible between the sets*

$$\{(x_i)_{i=1}^n \in G'_{[1,n]} : x_n = a\} \text{ and } \{(x_i)_{i=1}^n \in G'_{[1,n]} : x_n = b\},$$

then $\{X_i, f_i\}$ has fpp.

Proof. Let $H \subseteq \varprojlim \{X_i, f_i\}$ with $\pi_i(H) = X_i$ for infinitely many $i \in \mathbb{N}$. Let $j \in \mathbb{N}$ such that $\pi_j(H) = X_j$. Choose $a, b \in X_j$ such that $G'_{[1,j]}$ is irreducible between the sets

$$\{(x_i)_{i=1}^j \in G'_{[1,j]} : x_j = a\} \text{ and } \{(x_i)_{i=1}^j \in G'_{[1,j]} : x_j = b\}.$$

Since $\pi_j(H) = X_j$, it contains both a and b , so the continuum $\pi_{[1,j]}(H)$ must intersect both of

$$\{(x_i)_{i=1}^j \in G'_{[1,j]} : x_j = a\} \text{ and } \{(x_i)_{i=1}^j \in G'_{[1,j]} : x_j = b\}.$$

Therefore $\pi_{[1,j]}(H) = G'_{[1,j]}$, and hence $\pi_{[1,k]}(H) = G'_{[1,k]}$ for all $1 \leq k \leq j$.

Since this holds for all $j \in \mathbb{N}$ for which $\pi_j(H) = X_j$, and there are infinitely many such j , it follows that $\pi_{[1,k]}(H) = G'_{[1,k]}$ for all $k \in \mathbb{N}$. Therefore, $H = \varprojlim \{X_i, f_i\}$. \square

Corollary 23. *Let $\{X_i, f_i\}$ be an inverse sequence where for each i , $f_i : X_{i+1} \rightarrow 2^{X_i}$ is usc. Suppose that for each $n \in \mathbb{N}$, $G'_{[1,n]}$ is an arc, and whenever $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ are the end points of $G'_{[1,n]}$, $f_i(a_{i+1}) = \{a_i\}$ and $f_i(b_{i+1}) = \{b_i\}$ for each $1 \leq i < n$. Then $\varprojlim \{X_i, f_i\}$ is a continuum with fpp.*

Proof. We need only check that such an inverse sequence will satisfy the conditions of Theorem 22. Notice that since each $G'_{[1,n]}$ is an arc, Theorem 8 guarantees that $\varprojlim \{X_i, f_i\}$ will be a continuum.

Given $n \in \mathbb{N}$, let $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ be the endpoints of $G'_{[1,n]}$. By assumption, $f_i(a_{i+1})$ and $f_i(b_{i+1})$ are single valued for each $1 \leq i < n$. Therefore, the only point in $G'_{[1,n]}$ which has a_n as the n th coordinate is $(a_i)_{i=1}^n$, and likewise for b_n , so since $G'_{[1,n]}$ is irreducible between its endpoints, it will be irreducible between the sets

$$\{(x_i)_{i=1}^n \in G'_{[1,n]} : x_n = a_n\} \text{ and } \{(x_i)_{i=1}^n \in G'_{[1,n]} : x_n = b_n\}.$$

□

Corollary 24. *Let $f : [0, 1] \rightarrow 2^{[0,1]}$ be usc with the property that $f(0)$ and $f(1)$ are both single-valued, and $f(0), f(1) \in \{0, 1\}$. Then if for each $n \in \mathbb{N}$, $G'_{[1,n]}$ is an arc with endpoints $(a_i)_{i=1}^n$ and $(b_i)_{i=1}^n$ where $a_i, b_i \in \{0, 1\}$ for each $i = 1, \dots, n$, then $\varprojlim f$ is a continuum with fpp.*

The following lemma appears in [3].

Lemma 25. [3] *Suppose $(f_i)_{i=1}^\infty$ is a sequence of set-valued functions such that $f_i : [0, 1] \rightarrow 2^{[0,1]}$ for each positive integer i . If $n \in \mathbb{N}$, then $G'_{[1,n+1]} = \{(x_i)_{i=1}^{n+1} \in [0, 1]^{n+1} : (x_i)_{i=1}^n \in G'_{[1,n]} \text{ and } x_{n+1} \in f_n^{-1}(x_n)\}$.*

Lemma 26. *Let $g_1, \dots, g_k : [0, 1] \rightarrow [0, 1]$ be continuous functions such that*

- (1) $g_1(0) = 0$ and $g_1(x) > 0$ for all $0 < x \leq 1$,
- (2) $g_k(1) = 1$ and $g_k(x) < 1$ for all $0 \leq x < 1$, and
- (3) for odd j , $g_j(1) = g_{j+1}(1)$, and $g_j(x) < g_{j+1}(x)$ for all $0 \leq x < 1$, and for even j , $g_j(0) = g_{j+1}(0)$, and $g_j(x) < g_{j+1}(x)$ for all $0 < x \leq 1$,

then if $f = (g_1 \cup \dots \cup g_k)^{-1}$, then $f : [0, 1] \rightarrow 2^{[0,1]}$ is usc, and for each $n \in \mathbb{N}$, $G'_{[1,n]}$ is an arc with endpoints $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. Thus $\varprojlim f$ is a continuum with fpp.

Proof. For each $1 \leq j \leq k$, $G(g_j)$ is an arc with endpoints $(0, g_j(0))$ and $(1, g_j(1))$. Moreover, for each $1 \leq i, j \leq k$, $G(g_i) \cap G(g_j) \neq \emptyset$ if and only if $|i - j| \leq 1$, and $G(g_j) \cap G(g_{j+1}) = (1, g_j(1))$ for odd j and $G(g_j) \cap G(g_{j+1}) = (0, g_j(0))$ for even j . Thus $G(g_1 \cup \dots \cup g_k)$ is an arc from $(0, 0)$ to $(1, 1)$, and since $G'_{[1,2]}(f) = G(g_1 \cup \dots \cup g_k)$, it is an arc from $(0, 0)$ to $(1, 1)$ as well.

Suppose that $G'_{[1,n]}$ is an arc from $(0, 0, \dots, 0)$ to $(1, 1, \dots, 1)$ for some $n \in \mathbb{N}$. By Lemma 25, $G'_{[1,n+1]} = \{(x_i)_{i=1}^{n+1} \in [0, 1]^{n+1} : (x_i)_{i=1}^n \in G'_{[1,n]} \text{ and } x_{n+1} \in f_n^{-1}(x_n)\}$. Now $f^{-1} = g_1 \cup \dots \cup g_k$, so define for each $j = 1, \dots, k$, the function $h_j : G'_{[1,n]} \rightarrow G'_{[1,n+1]}$ by $h_j((x_i)_{i=1}^n) = (x_1, \dots, x_n, g_j(x_n))$, then each h_j is a homeomorphism onto its image, and $G'_{[1,n+1]} = h_1(G'_{[1,n]}) \cup \dots \cup h_k(G'_{[1,n]})$.

Since each h_j is a homeomorphism, each $h_j(G'_{[1,n]})$ is an arc, and its endpoints are $(0, \dots, 0, g_j(0))$ and $(1, \dots, 1, g_j(1))$, and $h_{j+1}(G'_{[1,n]})$ is an arc whose endpoints

are $(0, \dots, 0, g_{j+1}(0))$ and $(1, \dots, 1, g_{j+1}(1))$. Then from property 2, if j is odd, these arcs will intersect only at their common endpoint

$$(1, \dots, 1, g_{j+1}(1)) = (1, \dots, 1, g_j(1)),$$

and if j is even then the arcs will intersect only at their common endpoint

$$(0, \dots, 0, g_{j+1}(0)) = (0, \dots, 0, g_j(0)).$$

Thus, $G'_{[1, n+1]}$ is an arc with endpoints $(0, 0, \dots, 0)$ and $(1, 1, \dots, 1)$. Thus, by induction, the hypotheses of Corollary 24 are met, and so $\varprojlim f$ is a continuum with fpp. \square

In this last theorem, notice that if Condition 2 were changed to say that $g_k(0) = 1$ and $g_k(x) < 1$ for all $0 < x \leq 1$, then nothing would change except that $G'_{[1, n]}$ would instead be an arc from $(0, 0, \dots, 0)$ to $(0, 0, \dots, 0, 1)$. Or, if Conditions 1 and 2 were changed so that $g_1(1) = 0$ and $g_k(0) = 1$, then for even n , $G'_{[1, n]}$ would be an arc from $(0, 1, 0, 1, \dots, 0, 1)$ to $(1, 0, 1, 0, \dots, 1, 0)$, and for odd n , it would be an arc from $(0, 1, 0, 1, \dots, 0)$ to $(1, 0, 1, 0, \dots, 1)$. Lastly, if $g_1(1) = 0$ and $g_k(1) = 1$, then each $G'_{[1, n]}$ would be an arc from $(1, 1, \dots, 1, 0)$ to $(1, 1, \dots, 1)$.

Also, notice that Condition 3 in the theorem is equivalent to saying that the graph of $g_1 \cup \dots \cup g_k$ is an arc from $(0, 0)$ to $(1, 1)$.

In light of these observations, we have the following more general version of the theorem.

Theorem 27. *Let $a, b \in \{0, 1\}$ (not necessarily distinct), and let $g_1, \dots, g_k : [0, 1] \rightarrow [0, 1]$ be continuous functions such that*

- (1) $g_1(a) = 0$ and $g_1(x) > 0$ for all $x \neq a$,
- (2) $g_k(b) = 1$ and $g_k(x) < 1$ for all $x \neq b$, and
- (3) *the graph of $g_1 \cup \dots \cup g_k$ is an arc from $(a, 0)$ to $(b, 1)$,*

then if $f = (g_1 \cup \dots \cup g_k)^{-1}$, then for each $n \in \mathbb{N}$, $G'_{[1, n]}$ is an arc, and $\varprojlim f$ is a continuum with fpp.

The proof of this theorem would consist of separate cases for each combination of a and b in $\{0, 1\}$, and the proof of each case would be virtually identical to the proof of Lemma 26.

The following theorem is a generalization of the above results. It will allow for the construction of more complicated usc functions with fpp.

Theorem 28. *Let $\Lambda \subset [0, 1]$ be a closed set containing 0 and 1. Let Λ' be the set of limit points of Λ , and suppose that $\Lambda \setminus \Lambda'$ is dense in Λ . Let $\{g_\lambda\}_{\lambda \in \Lambda}$ be a collection of continuous functions such that for each $\lambda \in \Lambda$, $g_\lambda : [0, 1] \rightarrow [0, 1]$. Suppose the collection of functions satisfies the following:*

- (1) $0 \in g_\lambda([0, 1])$ if and only if $\lambda = 0$ and $1 \in g_\lambda([0, 1])$ if and only if $\lambda = 1$,
- (2) if $g_0^{-1}(0)$ either contains more than one element or contains any element other than 0 or 1, then 0 is a limit point of Λ ,
- (3) if $g_1^{-1}(1)$ either contains more than one element or contains any element other than 0 or 1, then 1 is a limit point of Λ ,
- (4) if $\lambda, \mu \in \Lambda$ with $\lambda < \mu$, then $g_\lambda(x) < g_\mu(x)$ for all $x \in (0, 1)$, and $G(g_\lambda) \cap G(g_\mu) \neq \emptyset$ if and only if $(\lambda, \mu) \cap \Lambda = \emptyset$,
- (5) if $(\lambda_i)_{i \in \mathbb{N}}$ is a sequence of points in Λ and $\lambda_i \rightarrow \mu$ as $i \rightarrow \infty$, then $g_{\lambda_i} \rightarrow g_\mu$ pointwise as $i \rightarrow \infty$.

Then if $f = \bigcup_{\lambda \in \Lambda} g_\lambda^{-1}$, then $\varprojlim f$ is a continuum with fpp.

Proof. From Theorem 6, we have that $\varprojlim f$ will be a continuum if and only if $\varprojlim f^{-1}$ is a continuum, and from Theorem 5 we have that $\varprojlim f^{-1}$ will be a continuum if and only if $G(f^{-1}) = G(\bigcup_{\lambda \in \Lambda} g_\lambda)$ is connected. To see that $G(f^{-1})$ is connected, suppose that $G(f^{-1}) = A \cup B$ where A and B are disjoint non-empty closed sets. Since for each $\lambda \in \Lambda$, $G(g_\lambda)$ is connected, each $G(g_\lambda)$ must be entirely contained in either A or B , so let $\mathcal{A} = \{\lambda \in \Lambda : G(g_\lambda) \subset A\}$ and $\mathcal{B} = \{\lambda \in \Lambda : G(g_\lambda) \subset B\}$. If there exists a separation of $G(f^{-1})$ then there exists a separation so that g_0 is separated from g_1 , so without loss of generality, suppose that $0 \in \mathcal{A}$ and $1 \in \mathcal{B}$.

Since each g_λ is continuous, then with Property 5, if $g : \Lambda \times [0, 1] \rightarrow [0, 1] \times [0, 1]$ is defined by $g(\lambda, x) = (x, g_\lambda(x))$, then g is continuous. In particular, $g^{-1}(A) = \mathcal{A} \times [0, 1]$, and $g^{-1}(B) = \mathcal{B} \times [0, 1]$ are closed sets (and disjoint), so \mathcal{A} and \mathcal{B} are closed and disjoint. Since they are both subsets of the compact set $[0, 1]$, they are also compact, so there exists a maximum element $\alpha \in \mathcal{A}$ and a minimum element $\beta \in \mathcal{B} \cap (\alpha, 1]$. Thus the interval (α, β) does not intersect Λ , which from Property 4 implies that $g_\alpha(x) = g_\beta(x)$ for some $x \in [0, 1]$, which in turn implies that $A \cap B \neq \emptyset$ which is a contradiction. Therefore, $G(f)$ (and hence $\varprojlim f$) is connected.

Now, to appeal to Theorem 22, it must first be shown that $G'_{[1,2]}$ is irreducible. Let $(a, 0)$ and $(b, 1)$ be points in $G'_{[1,2]}$. Suppose that K is a proper subcontinuum of $G'_{[1,2]}$ containing both $(a, 0)$ and $(b, 1)$. Then there exists $\lambda_0 \in \Lambda$, $z_0 \in [0, 1]$ with $(z_0, g_{\lambda_0}(z_0)) \notin K$.

Case 1: If $\lambda_0 \notin \Lambda'$, then $G(g_{\lambda_0})$ is an arc. Also, $0 < g_{\lambda_0}(z_0) < 1$. This is because, $g_{\lambda_0}(z_0)$ could only possibly be zero if $\lambda_0 = 0$, but then $g_0^{-1}(0)$ would contain multiple points meaning $0 \in \Lambda'$. We are supposing $\lambda \notin \Lambda'$ however, so $g_{\lambda_0}(z_0) > 0$. Similarly, $g_{\lambda_0}(z_0) < 1$. Then from Property 4, in order for K to contain both $(a, 0)$ and $(b, 1)$ —a point whose second coordinate is less than $g_{\lambda_0}(z_0)$ and a point whose second coordinate is greater than $g_{\lambda_0}(z_0)$ —then K must contain both endpoints of $G(g_{\lambda_0})$. This then means that K must contain $(z_0, g_{\lambda_0}(z_0))$, and we have a contradiction.

Case 2: If $\lambda_0 \in \Lambda'$, then since $\Lambda \setminus \Lambda'$ is dense in Λ , there exists a sequence $(\lambda_i)_{i \in \mathbb{N}}$ in $\Lambda \setminus \Lambda'$ converging to λ_0 , so $(z_0, g_{\lambda_i}(z_0)) \rightarrow (z_0, g_{\lambda_0}(z_0))$ as $i \rightarrow \infty$. Since $(z_0, g_{\lambda_0}(z_0)) \notin K$ and K is closed, there exists a neighborhood U of $(z_0, g_{\lambda_0}(z_0))$ which is disjoint from K . Thus, there exists some $j \in \mathbb{N}$ for which $\lambda_j \notin \Lambda'$ and $(z_0, g_{\lambda_j}(z_0)) \notin K$, which then from case 1 yields a contradiction.

As we have a contradiction in either case, no such K exists, so $G'_{[1,2]}$ is irreducible between the sets

$$\{(x_1, x_2) \in G'_{[1,2]} : x_2 = 0\} \text{ and } \{(x_1, x_2) \in G'_{[1,2]} : x_2 = 1\}.$$

Now, fix $n \in \mathbb{N}$, and suppose that $G'_{[1,n]}$ is irreducible between the sets

$$\{(x_i)_{i=1}^n \in G'_{[1,n]} : x_n = 0\} \text{ and } \{(x_i)_{i=1}^n \in G'_{[1,n]} : x_n = 1\}.$$

For each $\lambda \in \Lambda$, define $h_\lambda : G'_{[1,n]} \rightarrow G'_{[1,n+1]}$ by

$$h_\lambda((x_i)_{i=1}^n) = (x_1, x_2, \dots, x_n, g_\lambda(x_n)).$$

The map h_λ is a continuous injection, and if we define $G_\lambda = h_\lambda(G'_{[1,n]})$, by Lemma 25,

$$G'_{[1,n+1]} = \bigcup_{\lambda \in \Lambda} G_\lambda$$

As in the proof of Lemma 26, each h_λ is a homeomorphism onto its range, and hence, for each λ , G_λ is irreducible between the sets

$$G_\lambda(0) = \{(x_i)_{i=1}^{n+1} \in G'_{[1,n+1]} : x_n = 0, x_{n+1} = g_\lambda(0)\}$$

and

$$G_\lambda(1) = \{(x_i)_{i=1}^{n+1} \in G'_{[1,n+1]} : x_n = 1, x_{n+1} = g_\lambda(1)\}.$$

It is worth noting that these sets are $\pi_n^{-1}(0) \cap G_\lambda$ and $\pi_n^{-1}(1) \cap G_\lambda$ respectively where $\pi_n : G'_{n+1} \rightarrow [0, 1]$ is projection onto the n th coordinate.

Now, suppose that $K \leq G'_{[1,n+1]}$ intersects both $\{(x_i)_{i=1}^{n+1} \in G'_{[1,n+1]} : x_{n+1} = 0\}$ and $\{(x_i)_{i=1}^{n+1} \in G'_{[1,n+1]} : x_{n+1} = 1\}$, and suppose that $p \in G'_{[1,n+1]} \setminus K$. Then there exists a $\lambda_0 \in \Lambda$ and $z_0 \in G'_{[1,n]}$ with $h_{\lambda_0}(z_0) = p$.

Now, consider $K_0 = \pi_{[n,n+1]}(K)$. Since K is connected, K_0 is a connected subset of $G'_{[n,n+1]} = G'_{[1,2]}$. In particular, K_0 contains a point with second coordinate 0 and a point with second coordinate 1, and so from the base step of the induction, we know that $K_0 = G'_{[1,2]}$. This means that K_0 contains the points $(0, g_{\lambda_0}(0))$ and $(1, g_{\lambda_0}(1))$, which in turn, implies that K intersects both $G_{\lambda_0}(0)$ and $G_{\lambda_0}(1)$. As earlier, there are two cases.

Case 1: Suppose that $\lambda_0 \notin \Lambda'$. Then $G_{\lambda_0} \setminus \pi_n^{-1}(\{0, 1\})$ is open in $G'_{[1,n+1]}$, and so $[G_{\lambda_0} \setminus \pi_n^{-1}(\{0, 1\})] \cap K$ is open in K . Since K is connected, by the Boundary Bumping Theorem [6] the closure of each component of $[G_{\lambda_0} \setminus \pi_n^{-1}(\{0, 1\})] \cap K$ meets the boundary of $[G_{\lambda_0} \setminus \pi_n^{-1}(\{0, 1\})] \cap K$. But the boundary of this set in K is precisely $[G_{\lambda_0}(0) \cup G_{\lambda_0}(1)] \cap K$.

In particular, the closure of each component of $G_{\lambda_0} \setminus \pi_n^{-1}(\{0, 1\}) \cap K$ meets at least one of $G_{\lambda_0}(0)$ and $G_{\lambda_0}(1)$. However, at least one such component must meet both $G_{\lambda_0}(0)$ and $G_{\lambda_0}(1)$, else we would have a separation of K . Let L be one such component. Then the closure of L would be a subcontinuum of G_{λ_0} which meets both $G_{\lambda_0}(0)$ and $G_{\lambda_0}(1)$. Since G_{λ_0} is irreducible between these two sets, it follows that $L = G_{\lambda_0}$, and so $p \in G_{\lambda_0} = L \subseteq K$, contradicting our assumption that $p \notin K$.

Case 2: If $\lambda_0 \in \Lambda'$, let $U \ni p$ be an open subset of $G'_{[1,n+1]}$ which does not meet K . Proceeding similarly to Case 2 earlier, since U is open and $\lambda_0 \in \Lambda'$, there exists $\lambda_1 \in \Lambda \setminus \Lambda'$ with $p_1 = g_{\lambda_1}(z_0) \in U$. In particular, $p_1 \notin K$, so applying the argument from Case 1 to p_1 yields a contradiction.

Thus, K is not proper, and $G'_{[1,n+1]}$ must be irreducible between the sets

$$\{(x_i)_{i=1}^n \in G'_{[1,n+1]} : x_{n+1} = 0\} \text{ and } \{(x_i)_{i=1}^{n+1} \in G'_{[1,n+1]} : x_{n+1} = 1\}.$$

Thus, by induction, $G'_{[1,n]}$ satisfies the hypotheses of Theorem 22 for all $n \in \mathbb{N}$, so $\varprojlim f$ has fpp. \square

4. EXAMPLES

The purpose of this section is to demonstrate the utility of Theorem 19. To do so, we will provide examples of usc functions on the interval $[0, 1]$ which yield inverse limits.

Example 29. Let $f : [0, 1] \rightarrow 2^{[0,1]}$ with graph given as in Figure 1 in Section 2.

As already observed, f is an indecomposable usc function. By Theorem 27, $\varprojlim\{[0, 1], f\}$ is a continuum with fpp. Thus by Theorem 19, $\varprojlim\{[0, 1], f\}$ is indecomposable.

Notice that f does not satisfy the two-pass condition, as there are no connected open sets on which f is single valued which map onto $[0, 1]$. Notice also, that f is not a steeple, and thus neither Theorem 12 nor Theorem 13 apply to this example. The following is another example of a usc function with indecomposable inverse limit which does not satisfy the two-pass condition and is not a steeple.

Example 30. Let $g : [0, 1] \rightarrow 2^{[0,1]}$ with graph given as in Figure 2.

It is not difficult to see that g is an indecomposable usc function. In fact, the argument is identical to those seen in Example 16 and Lemma 18. Theorem 27 again gives us that $\varprojlim\{[0, 1], g\}$ is a continuum and has fpp. Once again, Theorem 19 tells us that it is indecomposable as well.

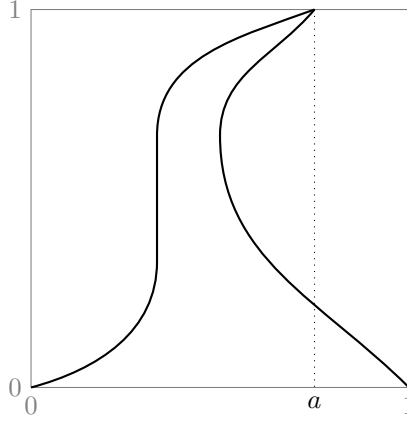


FIGURE 2. The graph of a usc function g on $[0, 1]$ with indecomposable inverse limit

The following examples are constructed using Theorem 28.

Example 31. Let $h : [0, 1] \rightarrow 2^{[0,1]}$ with graph given as in Figure 3, i.e. the product of the Cantor set and $[0, 1]$, along with curves from the top of the left endpoint of each removed interval to the bottom of its right endpoint. (Note that these curves must be inverses of graphs of functions).

It is not difficult to see that h is an indecomposable usc function. Again, the argument is very similar to the previous arguments. Theorem 28 (with Λ being the union of the Cantor set and the midpoints of the deleted intervals) gives us that $\varprojlim\{[0, 1], h\}$ is a continuum and has fpp. Once again, Theorem 19 tells us that it is indecomposable as well.

This next example is particularly interesting as it is nowhere single-valued.

Example 32. Let $p : [0, 1] \rightarrow 2^{[0,1]}$ with graph given as in Figure 4, i.e. the union of countably many piecewise linear functions which limit to the arcs $(0, 1)$, $(0, \frac{1}{2})$, $(\frac{1}{2}, 0)$

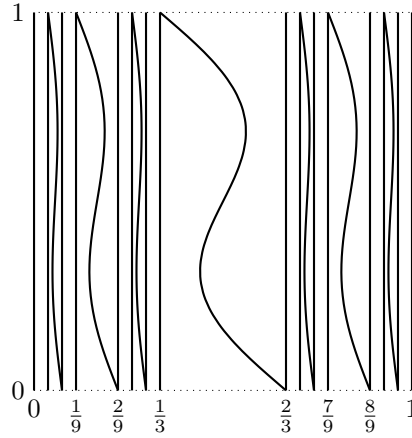


FIGURE 3. The graph of a usc function h on $[0, 1]$ with indecomposable inverse limit

and $(1, 0), (1, \frac{1}{2}), (\frac{1}{2}, 1)$. Notice that this usc function is nowhere single-valued. The continuum $\varprojlim \{[0, 1], p\}$ is indecomposable.

It is not difficult to see that p is an indecomposable usc function. Again, the argument is very similar to the previous arguments. Theorem 28 again (in this case $\Lambda = \{0, 1\} \cup \{\frac{1}{n}, \frac{n-1}{n} : n \in \mathbb{N}\}$) gives us that $\varprojlim \{[0, 1], p\}$ is a continuum and has fpp. Once again, Theorem 19 tells us that it is indecomposable as well.

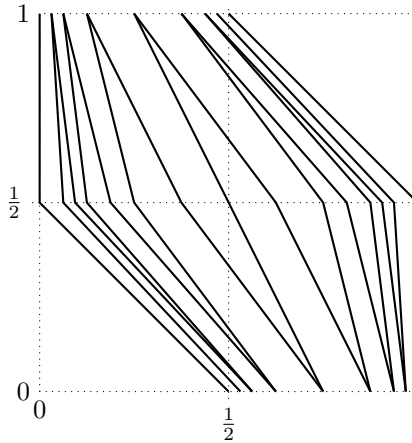


FIGURE 4. The graph of a usc function p on $[0, 1]$ with indecomposable inverse limit

REFERENCES

- [1] W. T. Ingram. Inverse limits of upper semi-continuous functions that are unions of mappings. *Topology Proc.*, 34:17–26, 2009.

- [2] W. T. Ingram. Inverse limits with upper semi-continuous bonding functions: problems and some partial solutions. *Topology Proc.*, 36:353–373, 2010.
- [3] W. T. Ingram. *An introduction to inverse limits with set-valued functions*. Springer Briefs in Mathematics. Springer, New York, 2012.
- [4] W. T. Ingram and William S. Mahavier. Inverse limits of upper semi-continuous set valued functions. *Houston J. Math.*, 32(1):119–130, 2006.
- [5] William S. Mahavier. Inverse limits with subsets of $[0, 1] \times [0, 1]$. *Topology Appl.*, 141(1-3):225–231, 2004.
- [6] Sam B. Nadler, Jr. *Continuum theory*, volume 158 of *Monographs and Textbooks in Pure and Applied Mathematics*. Marcel Dekker Inc., New York, 1992. An introduction.
- [7] Van Nall. Inverse limits with set valued functions. *Houston J. Math.*, 37(4):1323–1332, 2011.
- [8] Van Nall. Connected inverse limits with a set-valued function. *Topology Proc.*, 40:167–177, 2012.
- [9] Scott Varagona. Inverse limits with upper semi-continuous bonding functions and indecomposability. *Houston J. Math.*, 37(3):1017–1034, 2011.

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