An exploration of a mixed up-downwind scheme for solving Heston volatility model equations on variable grids

Chong Sun

jointly with Dr. Q. Sheng

Department of Mathematics
Center for Astrophysics, Space Physics and Engineering Research
Baylor University

January 13th, 2018

Outline:

Heston Stochastic Volatility Model

Heston proposed that volatility of stock return also follows a Brownian motion,

$$\begin{aligned} \mathrm{d}S(t) &= \mu S(t)\mathrm{d}t + S(t)\sqrt{y(t)}\mathrm{d}B(t) \\ \mathrm{d}y(t) &= \kappa[\eta - y(t)]\mathrm{d}t + \sigma\sqrt{y(t)}\mathrm{d}\tilde{B}(t) \\ \mathrm{d}B(t)\mathrm{d}\tilde{B}(t) &= \rho \mathrm{d}t \end{aligned}$$

where μ is the expected return of the underlying asset, κ is the rate of reversion to the mean level of volatility $y(t)$, η is the mean level that $y(t)$ reverse to and σ is the volatility of $y(t)$. Correlation coefficient is $\rho \in [-1, 1]$.

Heston Partial Differential Equation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

Heston Partial Differential Equation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

$$V(x, y, 0) = \max(1 - e^x, 0), \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^+,$$

Heston Partial Differential Equation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

$$\begin{aligned}
V(x,y,0) &= \max\left(1-e^x, 0\right), \quad x \in \mathbb{R}, \;\; y \in \mathbb{R}^+, \\
\lim_{x\to-\infty} V(x,y,\tau) &= 1, \quad y \in \mathbb{R}^+, \;\; \tau \in \mathbb{R}^+, \\
\lim_{x\to\infty} V(x,y,\tau) &= 0, \quad y \in \mathbb{R}^+, \;\; \tau \in \mathbb{R}^+, \\
V(x,0,\tau) &= \max\left(1-e^x, 0\right), \quad x \in \mathbb{R}, \;\; \tau \in \mathbb{R}^+.
\end{aligned}$$

Heston Partial Differential Equation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

$$\begin{aligned}
V(x,y,0) &= \max\left(1-e^x, 0\right), \quad x \in \mathbb{R}, \quad y \in \mathbb{R}^+, \\
\lim_{x\to-\infty} V(x,y,\tau) &= 1, \quad y \in \mathbb{R}^+, \quad \tau \in \mathbb{R}^+, \\
\lim_{x\to\infty} V(x,y,\tau) &= 0, \quad y \in \mathbb{R}^+, \quad \tau \in \mathbb{R}^+, \\
V(x,0,\tau) &= \max\left(1-e^x, 0\right), \quad x \in \mathbb{R}, \quad \tau \in \mathbb{R}^+. \\
\lim_{y\to\infty} V_y(x,y,\tau) &= 0, \quad x \in \mathbb{R}, \quad \tau \in \mathbb{R}^+,
\end{aligned}$$

Traditional Approaches and Limitation

Approaches:

- Central difference approximation

Traditional Approaches and Limitation

Approaches:

- Central difference approximation
- von Neumann method for stability analysis

Traditional Approaches and Limitation

Approaches:

- Central difference approximation
- von Neumann method for stability analysis

Limitation:

- von Neumann analysis can only be applied to Cauchy problems or periodic boundary conditions

Our Approach–Mixed Derivative

Mixed Derivative Term:

- Positive coefficient:

$$V_{xy}(x_m, y_n, \tau) \approx \frac{1}{2}(\Delta_{x,-}\Delta_{y,-} + \Delta_{x,+}\Delta_{y,+})V(x_m, y_n, \tau).$$

Our Approach–Mixed Derivative

Mixed Derivative Term:

- Positive coefficient:

$$V_{xy}(x_m, y_n, \tau) \approx \frac{1}{2}(\Delta_{x,-}\Delta_{y,-} + \Delta_{x,+}\Delta_{y,+})V(x_m, y_n, \tau).$$

- Negative coefficient:

$$V_{xy}(x_m, y_n, \tau) \approx \frac{1}{2}(\Delta_{x,+}\Delta_{y,-} + \Delta_{x,-}\Delta_{y,+})V(x_m, y_n, \tau).$$

Our Approach–Advection Terms

- Positive coefficient: Forward Difference Approximation
- Negative coefficient: Backward Difference Approximation

Semi-Discretised System

Semi-discretized system:

$$\mathbf{u}'(\tau) = \mathbf{M}\mathbf{u}(\tau) + \mathbf{f}(\tau),$$

The solution to (1) is

$$\mathbf{u}(\tau) = e^{\tau\mathbf{M}}\mathbf{u}(0) - \int_0^\tau e^{(\tau-s)\mathbf{M}}\mathbf{f}(s)ds.$$

Definition of Stability of Semi-Discretised Systems

Definition (Stability of Semi-Discretised Systems)

The semi-discretised system (1) is stable if for every $\tau^* > 0$, there exists a constant $c(\tau^*) > 0$ such that

$$\|e^{\tau \mathbf{M}}\| \leq c(\tau^*), \quad \tau \in [0, \tau^*]. \tag{1}$$

where $\|\cdot\|$ is an appropriate matrix norm.

Gerschgorin's Circle and Exponential Behavior Theorems

Theorem (Gerschgorin's Circle Theorem/Brauer's Theorem)

Let M_s be the sum of the moduli of the elements along the sth row of matrix $\boldsymbol{M}$ excluding the diagonal element m_{ss}. Then each eigenvalue of $\boldsymbol{M}$ lies inside or on the boundary of at least one of the circles $|\lambda - m_{ss}| = M_s$.

Gerschgorin's Circle and Exponential Behavior Theorems

Theorem (Gerschgorin's Circle Theorem/Brauer's Theorem)

Let M_s be the sum of the moduli of the elements along the sth row of matrix $\boldsymbol{M}$ excluding the diagonal element m_{ss}. Then each eigenvalue of $\boldsymbol{M}$ lies inside or on the boundary of at least one of the circles $|\lambda - m_{ss}| = M_s$.

Theorem (Exponential Behavior)

$e^{t\boldsymbol{A}}$ tends to 0 in certain norm hence in all norms, as t tends to $+\infty$, if and only if all the eigenvalues of $\boldsymbol{A}$ have strictly negative real parts.

Gerschgorin's Circle and Exponential Behavior Theorems

Theorem (Gerschgorin's Circle Theorem/Brauer's Theorem)

Let M_s be the sum of the moduli of the elements along the sth row of matrix $\boldsymbol{M}$ excluding the diagonal element m_{ss}. Then each eigenvalue of $\boldsymbol{M}$ lies inside or on the boundary of at least one of the circles $|\lambda - m_{ss}| = M_s$.

Theorem (Exponential Behavior)

$e^{t\boldsymbol{A}}$ tends to 0 in certain norm hence in all norms, as t tends to $+\infty$, if and only if all the eigenvalues of $\boldsymbol{A}$ have strictly negative real parts.

Theorem

For $\rho \in [-1, 1]$, the semi-discretised system (1) is stable.

Domain Truncation

$$V_\tau = \frac{1}{2} y V_{xx} + \rho\sigma y V_{xy} + \frac{1}{2}\sigma^2 y V_{yy} - \left(\frac{1}{2} y - r\right) V_x + \kappa(\eta - y) V_y,$$

$$\begin{aligned}
V(x, y, 0) &= \max\left(1 - e^x, 0\right), \quad x \in [-X,\ X], \quad y \in [0,\ Y], \\
V(-X, y, \tau) &= 1, \quad y \in [0,\ Y], \ \ \tau \in \mathbb{R}^+, \\
V(X, y, \tau) &= 0, \quad y \in [0,\ Y], \ \ \tau \in \mathbb{R}^+, \\
V(x, 0, \tau) &= \max\left(1 - e^x, 0\right), \quad x \in [-X,\ X], \quad \tau \in \mathbb{R}^+. \\
V_y(x, Y, \tau) &= 0, \quad x \in [-X,\ X], \ \ \tau \in \mathbb{R}^+,
\end{aligned}$$

Solution Surface

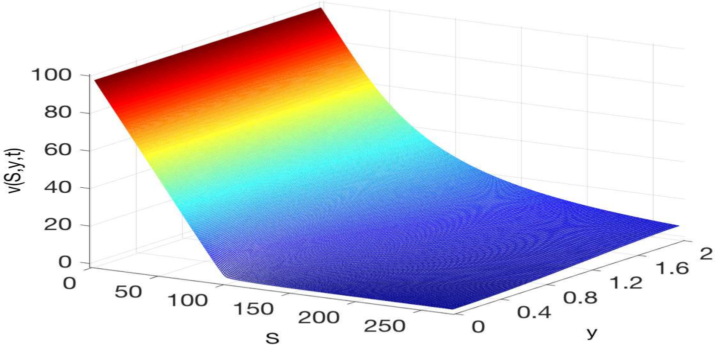

Figure: Price of an European put option

Convergence Surface

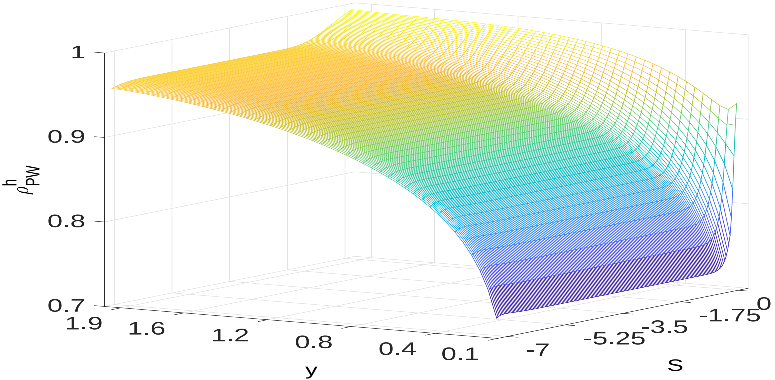

Figure: Rate of convergence ρ^h_{PW} surface at $T = 0.5$. The figure indicates approximately an order one rate of convergence.

Effect from Change of Scheme

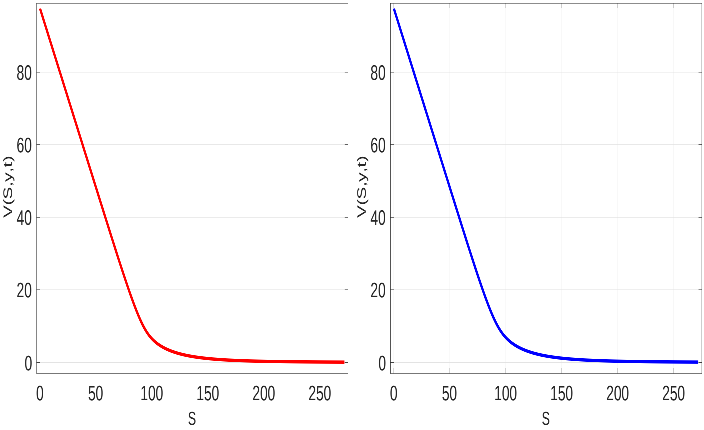

Figure: Comparison of change of schemes. Left: Before scheme change; Right: After scheme change.

Future Work

- Exponential Splitting and Padé Approximation
- Adaptive Grids
- Higher-Order Schemes
- American Options Pricing and Free Boundary Problems

Thank You