# A Stable Exponential Splitting Scheme For Pricing European Put Option Under Stochastic Volatility Model

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# Outline:

# Call Option vs Put Option:

- *Call Options* are contracts that give the holder the right to buy the underlying asset by a certain date for a certain price.
- *Put Options* are contracts that give the holder the right to sell the underlying asset by a certain date for a certain price.

# Terminologies for Options:

- *Maturity Date* of an option is the last day that the holder can exercise the option.
- *Strike Price* of an option is the price specified by the contract that the holder need to pay or can get when buying or selling the underlying asset.

# American Style vs European Style

- *American Options* can be exercised at any time up to the maturity date.
- *European Options* can be exercised only on the maturity date.

# Black-Scholes (BS) Model

Black-Scholes Model,

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sqrt{y}\mathrm{d}W(t) \tag{1}$$

where $S(t)$ is the stock price, $\mu$ is the drift term which represents the expected return on stock per year and $y$ is the fixed variance of stock price per year. $W(t)$ represents a Brownian motion process.

# Black-Scholes Partial Differential Equation

Let $v(S,t)$ be the price of an option. After applying Itô's lemma and non-arbitrage argument to (1), we can get rid of the stochastic factor and reach the Partial Differential Equation,

$$v_t + \frac{1}{2} y S^2 v_{SS} + rSv_S - rv = 0, \tag{2}$$

where $r$ is the fixed risk-less interest rate.

# Drawbacks

It is widely recognized that this classic option pricing model does not ideally fit empirical market data. Two identified empirical features have been under attention,

- Skewed distribution with higher peak and heavier tails of the return distribution;
- The volatility smile: A plot of the implied volatility of an option with certain life as a function of its strike price.

# Revisions

- Adding jumps to the model: Merson and Kou proposed jump-diffusion models with finite-jumps etc.;
- Stochastic volatility/interest rate: Heston considered stochastic volatility; Heston, Hull, White proposed stochastic volatility and stochastic interest rate;
- Combination of jumps with stochastic volatility proposed by David Bates.

# Heston Stochastic Volatility Model

Heston proposed that volatility of stock return also follows a Brownian motion,

$$\frac{\mathrm{d}S(t)}{S(t)} = \mu \mathrm{d}t + \sqrt{y(t)}\mathrm{d}W_1(t) \quad (3)$$

$$\mathrm{d}y(t) = \kappa(\eta - y(t))\mathrm{d}t + \sigma\sqrt{y(t)}\mathrm{d}W_2(t) \quad (4)$$

$$\mathrm{cov}(\mathrm{d}W_1(t), \mathrm{d}W_2(t)) = \rho \mathrm{d}t \quad (5)$$

where $\mu$ as before is the expected return of the underlying asset, $\kappa$ is the rate of reversion to the mean level of volatility $y(t)$, and $\sigma$ is the volatility of $y(t)$. Correlation coefficient of the two Brownian motions $W_1(t)$ and $W_2(t)$ is assumed to be $\rho \in [-1, 1]$.

# Heston Partial Differential Equation

Still by Itô's lemma and nonarbitrage argument, we reach the partial differential equation,

$$\begin{aligned} 0 &= v_t + \frac{1}{2} y S^2 \frac{\partial^2 v}{\partial S^2} + \rho\sigma y S \frac{\partial^2 v}{\partial S \partial y} + \frac{1}{2}\sigma^2 y \frac{\partial^2 v}{\partial y^2} \\ &\quad + rS\frac{\partial v}{\partial S} + \kappa(\eta - y)\frac{\partial v}{\partial y} - rv. \end{aligned} \tag{6}$$

# Initial Boundary Value Conditions

Here we consider European put options.
The payoff function for put option with strike price $K$ is

$$v(x, y, 0) = \max\{K - x, 0\}, \quad x \in \mathbf{R}^+, \ y \in \mathbf{R}^+. \tag{7}$$

We impose the following boundary conditions.

$$\begin{aligned} v(0, y, t) &= Ke^{-r(T-t)}, \quad y \in \mathbf{R}^+, \ t \in \mathbf{R}^+, \\ \lim_{x \to \infty} v(x, y, t) &= 0, \quad y \in \mathbf{R}^+, \ t \in \mathbf{R}^+, \\ v(x, 0, t) &= e^{-r(T-t)} \max\{K - x, 0\}, \quad x \in \mathbf{R}^+, \ t \in \mathbf{R}^+, \\ \lim_{y \to \infty} v_y(x, y, t) &= 0, \quad x \in \mathbf{R}^+, \ t \in \mathbf{R}^+. \end{aligned}$$

# Coordinate Transformation

In order to solve the PDE system effectively, we introduce a new variable as the time to expiration:$\tau = T - t$. And further we let $x = \ln\left(\frac{S}{K}\right)$ and $u = e^{r\tau}\frac{V}{K}$.

# Coordinate Transformation

$$u_\tau = \frac{1}{2}y\frac{\partial^2 u}{\partial x^2} + \rho\sigma y\frac{\partial^2 u}{\partial x \partial y} + \frac{1}{2}\sigma^2 y\frac{\partial^2 u}{\partial y^2} - \left(\frac{1}{2}y - r\right)\frac{\partial u}{\partial x} + \kappa(\eta - y)\frac{\partial u}{\partial y},$$

with the transformed initial boundary value conditions.

$$\begin{aligned}
u(x,y,0) &= K \cdot \max\left(1 - e^x, 0\right), \quad x \in \mathbf{R},\ y \in \mathbf{R}^+, \\
\lim_{x\to\infty} u(x,y,\tau) &= 0, \quad y \in \mathbf{R}^+,\ \tau \in \mathbf{R}^+, \\
\lim_{x\to-\infty} u(x,y,\tau) &= Ke^{-rt}, \quad y \in \mathbf{R}^+,\ \tau \in \mathbf{R}^+, \\
\lim_{y\to\infty} u_y(x,y,\tau) &= 0, \quad x \in \mathbf{R},\ \tau \in \mathbf{R}^+, \\
u(x,0,\tau) &= Ke^{-rt}\max\left(1 - e^x, 0\right), \quad x \in \mathbf{R},\ \tau \in \mathbf{R}^+.
\end{aligned}$$

# Traditional Approaches to Cross Derivatives

- Standard central difference $u_{xy} \approx \Delta_{x,0}\Delta_{y,0}u$ to discretise the cross derivative term;
- Compact difference scheme based on central difference to achieve fourth order approximation in space;
- von Neumann analysis for stability analysis.

# Limitations of Traditional Approaches

However, von Neumann analysis can only be applied to Cauchy problems, partial differential equations with periodic boundary conditions and certain Dirichlet boundary conditions.

# Our Approach

- First-order upwind-downwind scheme,
  - $u_{xy} \approx \Delta_{x,+}\Delta_{y,+}u$ for $\rho \leq 0$,
  - $u_{xy} \approx \Delta_{x,-}\Delta_{y,-}u$ for $\rho > 0$;
- Discretise the leading error term in the upwind-downwind scheme to achieve second-order approximation;
- Eigenvalue method for stability analysis.

# Domain Truncation

- For computational purpose:
  - Replace the domain of $x$ by $(-X, X)$ for large enough $X$,
  - Replace the domain of $y$ by $(0, 1)$ because of empirical evidence.
- Let $h_x$ denote the uniform mesh in the $x-$ direction and $h_y$ denote the uniform mesh in the $y-$ direction;
- We impose grids $\{x_m\}_0^{M+1}$ and $\{y_n\}_0^{N+1}$ in $x-$ and $y-$ directions respectively; We let $u_{m,n} = u(x_m, y_n)$.
- For stability analysis purpose, we let $h_y = \sigma h_x$.

# First-Order Discretisation of the Cross Derivative Terms

- When $\rho \in [-1, 0]$,

$$\begin{aligned} u_{xy}(x_m, y_n) &= \Delta_{x,+}\Delta_{y,+}u_{m,n} - \frac{h_y}{2}\frac{\partial^3 u_{m,n}}{\partial x \partial y^2} - \frac{h_x}{2}\frac{\partial^3 u_{m,n}}{\partial x^2 \partial y} + \mathcal{O}(h_x^2) \\ &= \frac{1}{h_x h_y}(u_{m,n} - u_{m,n+1} - u_{m+1,n} + u_{m+1,n+1}) \\ &\quad - \frac{h_y}{2}\frac{\partial^3 u_{m,n}}{\partial x \partial y^2} - \frac{h_x}{2}\frac{\partial^3 u_{m,n}}{\partial x^2 \partial y} + \mathcal{O}(h_x^2), \end{aligned} \tag{8}$$

# First-Order Discretisation of the Cross Derivative Terms Cont'd

- When $\rho \in [0,1]$,

$$\begin{aligned} u_{xy}(x_m, y_n) &= \Delta_{x,-}\Delta_{y,-}u_{m,n} + \frac{h_y}{2}\frac{\partial^3 u_{m,n}}{\partial x \partial y^2} - \frac{h_x}{2}\frac{\partial^3 u_{m,n}}{\partial x^2 \partial y} + \mathcal{O}(h_x^2) \\ &= \frac{1}{h_x h_y}(u_{m,n} - u_{m-1,n} - u_{m,n-1} + u_{m-1,n-1}) \\ &\quad + \frac{h_y}{2}\frac{\partial^3 u_{m,n}}{\partial x \partial y^2} + \frac{h_x}{2}\frac{\partial^3 u_{m,n}}{\partial x^2 \partial y} + \mathcal{O}(h_x^2) \end{aligned} \tag{9}$$

# Second Order Discretisation of the Cross Derivative Term

To get the second order discretisation, we use the following two first order discretisations to approximate the leading error terms.
For $\rho \in [-1, 0]$,

$$-\frac{h_y}{2}\frac{\partial^3 u_{m,n}}{\partial x \partial y^2} \approx -\frac{h_y}{2}\Delta_{x,+}\Delta^2_{y,0}u_{m,n}, \tag{10}$$

$$-\frac{h_x}{2}\frac{\partial^3 u_{m,n}}{\partial x^2 \partial y} \approx -\frac{h_x}{2}\Delta_{y,+}\Delta^2_{x,0}u_{m,n}. \tag{11}$$

# Second Order Discretisation of the Cross Derivative Term

For $\rho \in (0, 1]$,

$$\frac{h_y}{2}\frac{\partial^3 u_{m,n}}{\partial x \partial y^2} \approx \frac{h_y}{2}\Delta_{x,-}\Delta_{y,0}^2 u_{m,n}, \quad (12)$$

$$\frac{h_x}{2}\frac{\partial^3 u_{m,n}}{\partial x^2 \partial y} \approx \frac{h_x}{2}\Delta_{y,-}\Delta_{x,0}^2 u_{m,n}. \quad (13)$$

# Second Order Discretisation of the Cross Derivative Term Cont'd

- Here we only consider the case when $\rho \in [-1, 0]$.
- The generalization to $\rho \in (0, 1]$ is direct.

# Second Order Discretisation of the Cross Derivative Term Cont'd

Substitute (12) and (13) into (8),

$$
\begin{aligned}
u_{xy}(x_m, y_n) = & \frac{1}{h_x h_y}\Big(\frac{1}{2}u_{m-1,n} - \frac{1}{2}u_{m-1,n+1} + \frac{1}{2}u_{m,n-1} \\
& - u_{m,n} + \frac{1}{2}u_{m,n+1} - \frac{1}{2}u_{m+1,n-1} + \frac{1}{2}u_{m+1,n}\Big) \\
& + \mathcal{O}(h_x^2). \qquad (14)
\end{aligned}
$$

# Discretisation of Diffusion, Advection and Boundary Terms

- Standard central difference for the diffusion terms,

$$
\begin{aligned}
u_{xx}(x_m, y_n) &\approx \Delta^2_{x,0} u_{m,n} && (15)\\
u_{yy}(x_m, y_n) &\approx \Delta^2_{y,0} u_{m,n} && (16)
\end{aligned}
$$

- Central difference for the advection terms,

$$
\begin{aligned}
u_x(x_m, y_n) &\approx \Delta_{x,0} u_{m,n} && (17)\\
u_y(x_m, y_n) &\approx \Delta_{y,0} u_{m,n} && (18)
\end{aligned}
$$

- Central difference for the Neumann boundary condition,

$$
u_y(x_m, y_{N+1}) \approx \Delta_{y,0} u_{m,y_{N+1}} \qquad (19)
$$

# Semi-Discretised System

Substitute (8)-(19) into the transformed partial differential equation, we can get a semi-discretised system,

$$\mathbf{u}'(\tau) \quad = \quad \mathbf{M}\mathbf{u}(\tau) + \mathbf{f}, \tag{20}$$

- $\mathbf{u}(\tau) = \begin{bmatrix} u_{1,1} & u_{1,2} & \dots & u_{1,N+1} & u_{2,1} & \dots & u_{M,N+1} \end{bmatrix}^T$ is the $M(N+1) \times 1$ vector that contains all the grid points.
- $\mathbf{M}$ is the $M(N+1) \times M(N+1)$ matrix that contains the coefficients of grid points in the semi-discretised system.
- $\mathbf{f}$ is the $M(N+1) \times 1$ vector that is resulted from the nonhomogeneous Dirichlet boundary conditions and the Neumann boundary condition.

# Definition of Stability of Semi-Discretised Systems

The solution to (20) is

$$\mathbf{u}(\tau) = e^{\tau\mathbf{M}}\mathbf{u}(0) - \mathbf{M}^{-1}(\mathbf{I} - e^{\tau\mathbf{M}})\mathbf{f} \tag{21}$$

**Definition (Stability of Semi-Discretised Systems)**

The semi-discretised system (20) is stable if for every $\tau^* > 0$, there exists a constant $c(\tau^*) > 0$ such that

$$\|e^{\tau\mathbf{M}}\| \leq c(\tau^*), \quad \tau \in [0, \tau^*]. \tag{22}$$

where $\|\cdot\|$ is appropriate matrix norm.

# Gerschgorin's Circle and Exponential Behavior Theorems

**Theorem (Gerschgorin's Circle Theorem/Brauer's Theorem)**

*Let $M_s$ be the sum of the moduli of the elements along the sth row of matrix $\boldsymbol{M}$ excluding the diagonal element $m_{ss}$. Then each eigenvalue of $\boldsymbol{M}$ lies inside or on the boundary of at least one of the circles $|\lambda - m_{ss}| = M_s$. Moreover, an eigenvalue may lie on the boundary of one of the Gerschgorin's circles only if it lies on the boundaries of every Gerschgorin's circles.*

**Theorem (Exponential Behavior)**

*$e^{t\boldsymbol{A}}$ tends to 0 in certain norm hence in all norms, as t tends to $+\infty$ , if and only if all the eigenvalues of $\boldsymbol{A}$ have strictly negative real parts.*

# Stability Analysis

**Lemma**

*For $\rho \in [-1, 1]$, the semi-discretised system (20) is stable.*

Proof:
A general equation in the semi-discretised system without influence from boundary conditions is of the form,

$$\begin{aligned} u'_{m,n} &= a_{m,n}u_{m-1,n} - \frac{\rho\sigma y_n}{2h_x h_y}u_{m-1,n-1} + b_{m,n}u_{m,n-1} \\ &+ c_{m,n}u_{m,n} + d_{m,n}u_{m,n+1} - \frac{\rho\sigma y_n}{2h_x h_y}u_{m+1,n-1} \\ &+ p_{m,n}u_{m+1,n} \end{aligned} \tag{23}$$

# Proof of Stability

- $a_{m,n} = \frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} + \frac{1}{4h_x}(y_n - r)$
- $b_{m,n} = \frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} - \frac{\kappa(\eta - y_n)}{2h_y}$
- $c_{m,n} = -\frac{y_n}{h_x^2} - \frac{\sigma^2 y_n}{h_y^2} - \frac{\rho\sigma y_n}{h_x h_y}$
- $d_{m,n} = \frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} + \frac{\kappa(\eta - y_n)}{2h_y}$
- $p_{m,n} = \frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} - \frac{1}{4h_x}(y_n - r)$

# Proof Cont'd

The Gerschgorin circle corresponding to (23) is

$$
\begin{aligned}
\left|\lambda_{m,n} + \frac{y_n}{h_x^2} + \frac{\sigma^2 y_n}{h_y^2} + \frac{\rho\sigma y_n}{h_x h_y}\right| \leq & \left|\frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} + \frac{1}{4h_x}(y_n - r)\right| \\
& + \left|\frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} - \frac{\kappa(\eta - y_n)}{2h_y}\right| \\
& + \left|\frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} + \frac{\kappa(\eta - y_n)}{2h_y}\right| \\
& + \left|\frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} - \frac{1}{4h_x}(y_n - r)\right| - \frac{\rho\sigma y_n}{h_x h_y} \qquad (24)
\end{aligned}
$$

# Proof Cont'd

Let $x_{m,n} = Real(\lambda_{m,n})$. Then by triangle inequality of moduli, we have,

$$\begin{aligned}\left| x_{m,n} + \frac{y_n}{h_x^2} + \frac{\sigma^2 y_n}{h_y^2} + \frac{\rho\sigma y_n}{h_x h_y} \right| \leq & \left| \frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} + \frac{1}{4h_x}(y_n - r) \right| \\ & + \left| \frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} - \frac{\kappa(\eta - y_n)}{2h_y} \right| \\ & + \left| \frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} + \frac{\kappa(\eta - y_n)}{2h_y} \right| \\ & + \left| \frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} - \frac{1}{4h_x}(y_n - r) \right| - \frac{\rho\sigma y_n}{h_x h_y} \qquad (25)\end{aligned}$$

# Proof Cont'd

We use the fact that $h_y = \sigma h_x$,

$$\begin{aligned} \frac{y_n}{2h_x^2} + \frac{\rho\sigma y_n}{2h_x h_y} &= \frac{y_n}{2h_x^2}\left(1 + \frac{\rho\sigma h_y}{h_x}\right) \\ &= \frac{y_n}{2h_x^2}(1+\rho) \\ &\geq 0. \end{aligned} \tag{26}$$

# Proof Cont'd

And also,

$$\begin{aligned}\frac{\sigma^2 y_n}{2h_y^2} + \frac{\rho\sigma y_n}{2h_x h_y} &= \frac{\sigma y_n}{2h_y^2}\left(\sigma + \frac{\rho h_y}{h_x}\right) \\ &= \frac{\sigma^2 y_n}{2h_y^2}(1+\rho) \\ &\geq 0. \qquad (27)\end{aligned}$$

# Proof Cont'd

Thus for $h_x$ and $h_y$ small enough, we can get rid of all the absolute values on both sides of (25). After some simple algebraic manipulations, we have,

$$x_{m,n} \leq 0. \tag{28}$$

# Proof Cont'd

Now we can proceed to consider the influence of Neumann boundary condition.
A general equation with influence of Neumann boundary condition is of the form,

$$\begin{aligned} u'_{m,N+1} &= a_{m,N+1}u_{m-1,N+1} - \frac{\rho\sigma y_{N+1}}{2h_x h_y}u_{m-1,N} + q_{m,N+1}u_{m,N} \\ &+ c_{m,N+1}u_{m,N+1} - \frac{\rho\sigma y_n}{2h_x h_y}u_{m+1,N} \\ &+ p_{m,n}u_{m+1,N+1} \end{aligned} \tag{29}$$

where $q_{m,N+1} = \frac{\sigma^2 y_n}{h_y^2} + \frac{\rho\sigma y_n}{h_x h_y}$ while all other coefficients are defined same as before.

# Proof Cont'd

- By similar process, we can prove the Gerschgorin's circles corresponding to (29) also lies on the left half of complex plane.
- Because the Gerschgorin's circles corresponding to equations with influence from Dirichlet boundary condition will locate strictly on the left half of complex plane, we conclude that the real part of eigenvalues of **M** are strictly negative.
- $e^{\tau \mathbf{M}}$ is bounded in some norm and hence in all norms by the exponential behavior theorem. And thus the semi-discretised system is stable.

# Conclusion

- $e^{\tau \mathbf{M}}$ is bounded in some norm and hence in all norms by the exponential behavior theorem. And thus the semi-discretised system is stable.
- By same argument, we can generalize the result to when $\rho \in (0, 1]$ with the corresponding discretisation method.

# Future Work

- Exponential Splitting and Padé Approximation
- Numerical Experiment
- Nonuniform Grids
- Higher-Order Compact Schemes
- American Options Pricing and Free Boundary Problems
- More General Principle for the Discretisation of Cross Derivative Terms

# Questions?

Thank You